

MATH 110: Class 06

August 24: Spatial Representations of the mathematical objects, Map projections, Mathematics of Voting

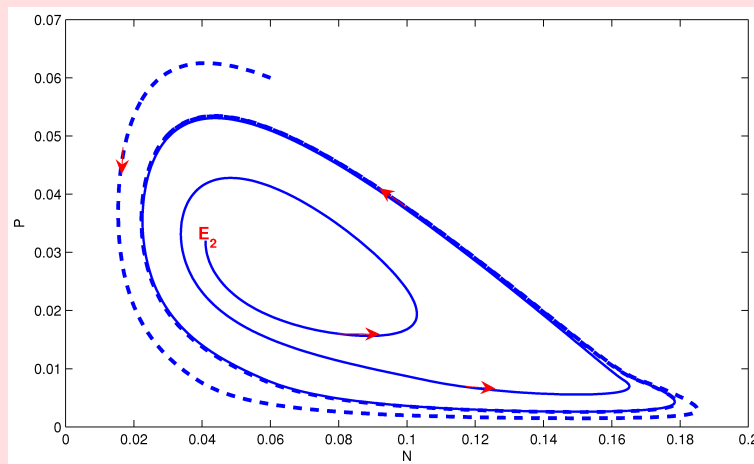
For today:

Homework Problem Set 2

Representing two pieces of numerical data at the same time: The Cartesian plane

1. **The Cartesian plane.** Often we are simultaneously interested in two numerical data at the same time—for example, if we are simultaneously studying the populations of two species, or recording both the horizontal and vertical displacement of an object. In many cases it is convenient to use the Cartesian plane, which is a visual representation of the pairs (a, b) of real numbers. We often denote the Cartesian plane by \mathbb{R}^2 , which is a shorthand notation for $\mathbb{R} \times \mathbb{R}$. Typically we label the horizontal axis the x -axis and the vertical the y -axis, so that (a, b) represents the situation wherein $x = a$ and $y = b$.

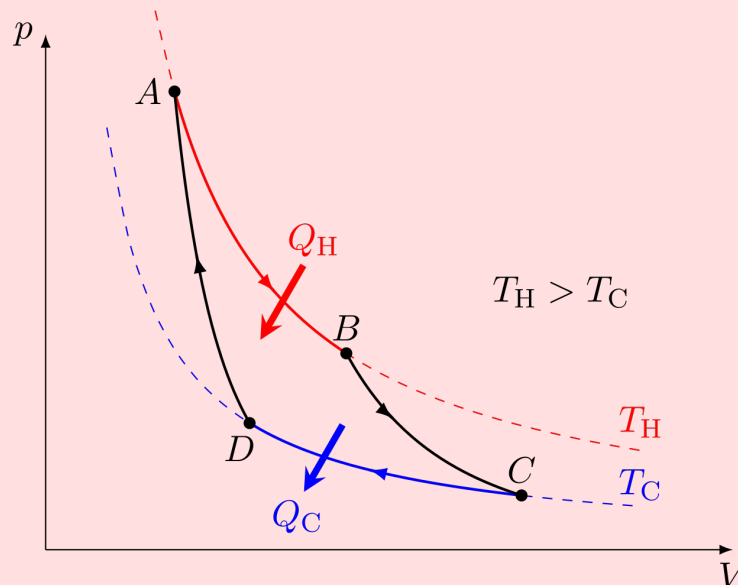
EXAMPLE (Population ecology: Predator–prey model with a stable limit cycle).



Li, B.; Liu, S.; Cui, J.; Li, J. A Simple Predator-Prey Population Model with Rich Dynamics. *Appl. Sci.* (2016) **6** 151. <https://www.mdpi.com/2076-3417/6/5/151>

This plot shows the evolution of a predator–prey model over time. Here, N is (in some units) the population of a prey species and P the population of the predator species. Notice that for both trajectories, corresponding to different starting populations of the two species, approach a stable loop—like the cycles in our (discrete dynamics) icebreaker activity. (Here E_2 marks an *unstable equilibrium*.)

EXAMPLE (Thermodynamics: Carnot cycle).



This plot shows the evolution of the volume (V) and pressure (P) of a fixed quantity of gas in a Carnot engine. The total amount of work done per cycle corresponds to the area of the region enclosed by the trajectory.

Map projections

1. The Earth's surface is (approximately) spherical, but for many purposes flat maps are more convenient than globes or sections thereof. So, we often define *map projections*, which are functions ϕ from a subset of the sphere to a subset of the plane. For any such projection, the point p on the Earth appears at the coordinates $\phi(p)$ in the flat plane.
2. **Question: What makes for a good map projection?**
3. **A little differential geometry: Curvature of curves and surfaces**
 - **Curvature of a curve.** The curvature $\kappa(s_0)$ of a curve $\gamma(s)$ at a point $\gamma(s_0)$ on the curve is the inverse of the radius of the osculating circle to γ at $\gamma(s_0)$, or zero if the osculating circle is degenerately a line.
 - **Cross sections and curvature.** Different cross-sections of a surface through a given point generally have different curvatures. If k_1 and k_2 are the maximum and minimum curvatures respectively, we call $\kappa = k_1 k_2$ the **Gaussian curvature** of the surface.

EXAMPLE (Surfaces of constant Gaussian curvature)

- (a) The Gaussian curvature of a sphere of radius R is R^{-2} at every points.
- (b) The Gaussian curvature of a plane is 0.
- (c) The Gaussian curvature of hyperbolic space (roughly speaking, and infinite saddle shape) is negative (and constant).

THEOREM. (Gauss's Theorema Egregium) If a function from one surface to another preserves distances, then it also preserves Gaussian curvature.

COROLLARY. Any map projection from (a piece of) a sphere to (a piece of) the plane cannot preserve distances.

- Put another way, the corollary shows that there is no such thing as a perfect map of a spherical world! So, when we use (or design) a map projection, we have to compromise the accuracy and proportionality of some features some features to maintain others.

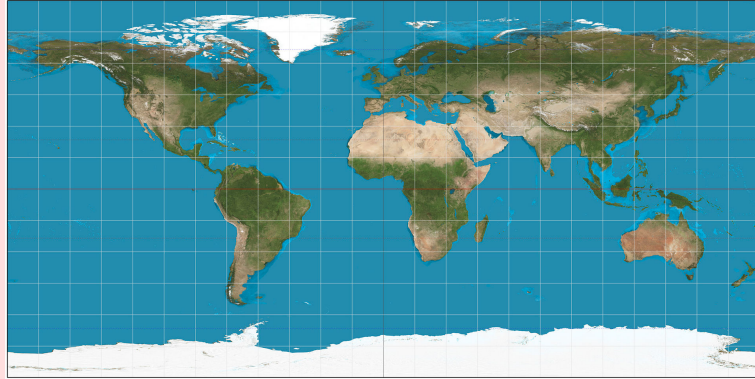
4. Map projections

There are many different map projections.

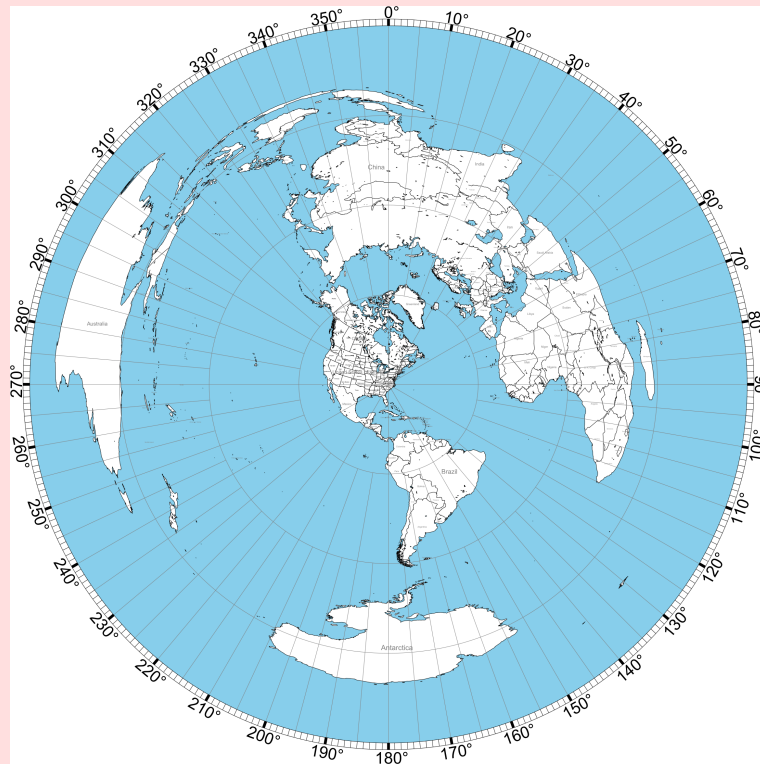
EXAMPLE.

Identify advantages and disadvantages of each projection.

- (a) **Equirectangular projection** (c. 120 C.E.). (This is actually an entire family of projections, depending on the choice of standard latitude/longitude. The choice of the equator gives a special case of equirectangular projection called **plate carrée**.)



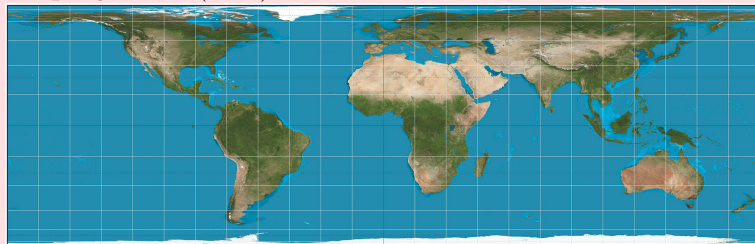
- (b) **Azimuthal equidistant projection** (11th Century). (Again there is an entire family of projections; we can choose any point for the center, from which distances are preserved; this map centers on Greensboro.)



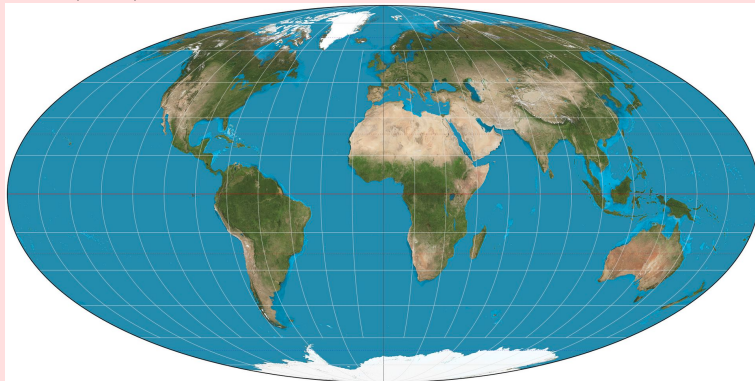
(c) **Mercator projection (1569).**



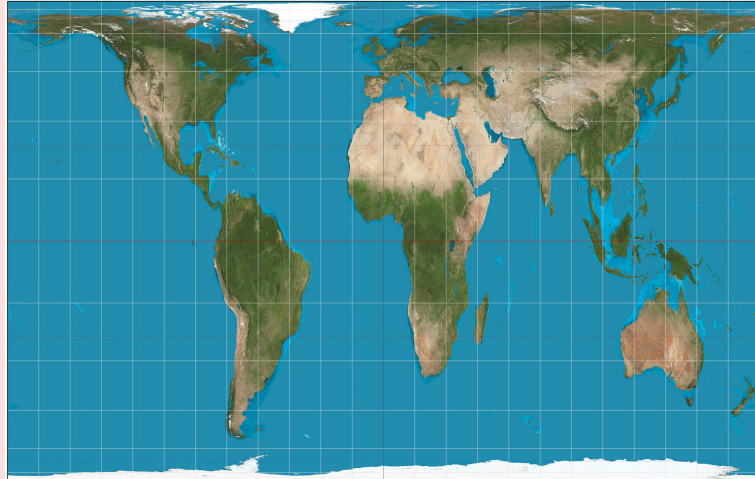
(d) **Lambert cylindrical projection (1772).**



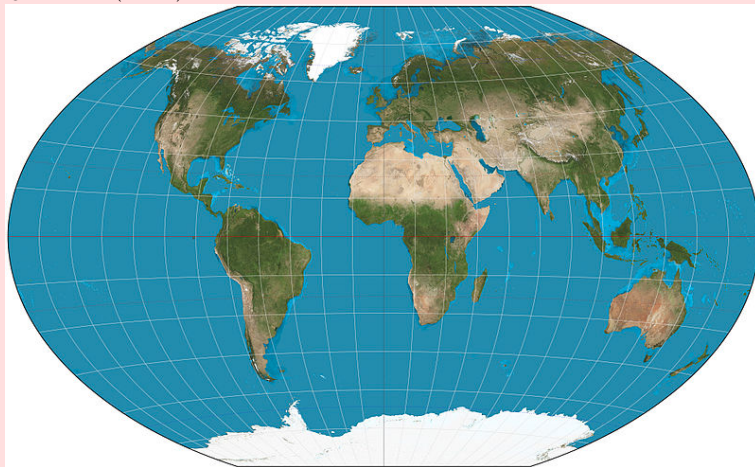
(e) **Mollweide projection (1805).**



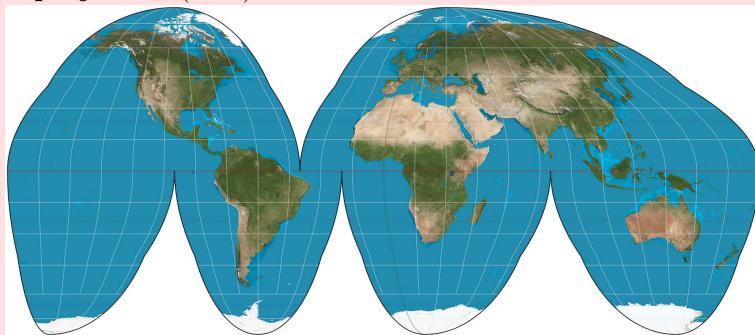
(f) **Gall–Peters projection (1855).**



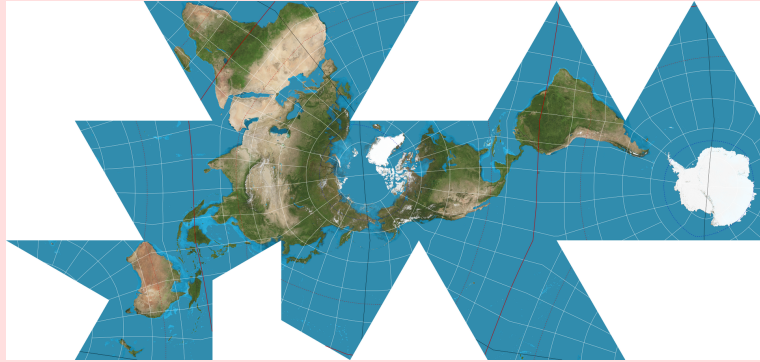
(g) **Winkel tripel projection (1921).**



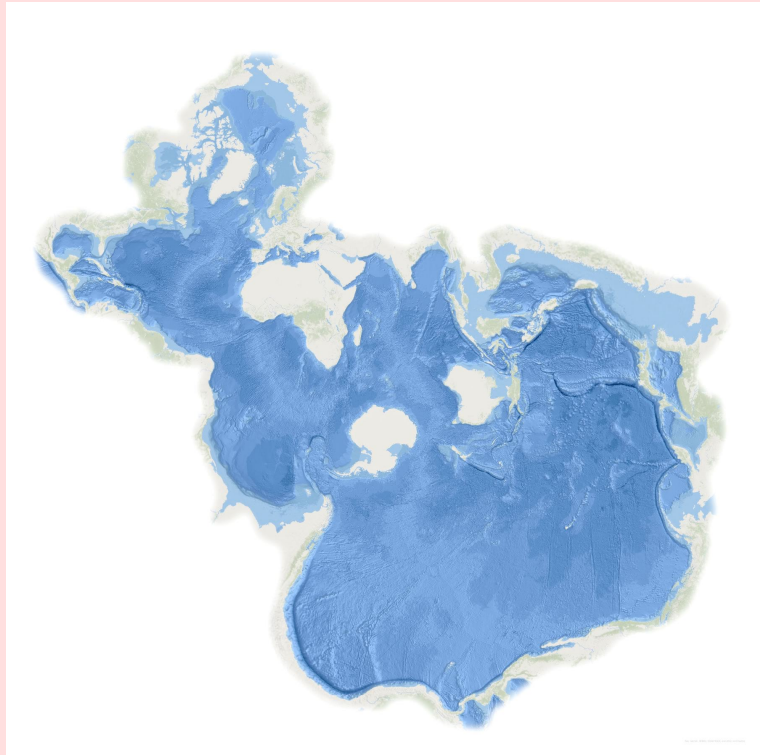
(h) **Goode homolosine projection (1923).**



(i) **Dymaxion map (1943).**



(j) **Spilhaus World Ocean map (this version 1979?).**



(k) **Waterman butterfly projection (1996).**



5. Some properties of a projection that are sometimes useful

- (a) Area preservation: The relative sizes of different regions (e.g., countries) are preserved.
- (b) Conformality: Preservation of angles; preservation of angles is guarantees that the map is approximately isometric in any small region.
- (c) Distance-preserve: Preserves some distances (the Theorema Egregium says that we can't preserve all, but see, e.g., the azimuthal equidistant projection).
- (d) Cardinal directions are the same at each point.
- (e) Conformality + cardinal directions are the same at each point guarantee that rhumb lines (lines of constant heading) are straight lines in the projection; the Mercator projection is essentially the only one that has the property.

6. Applications of unusual map projections

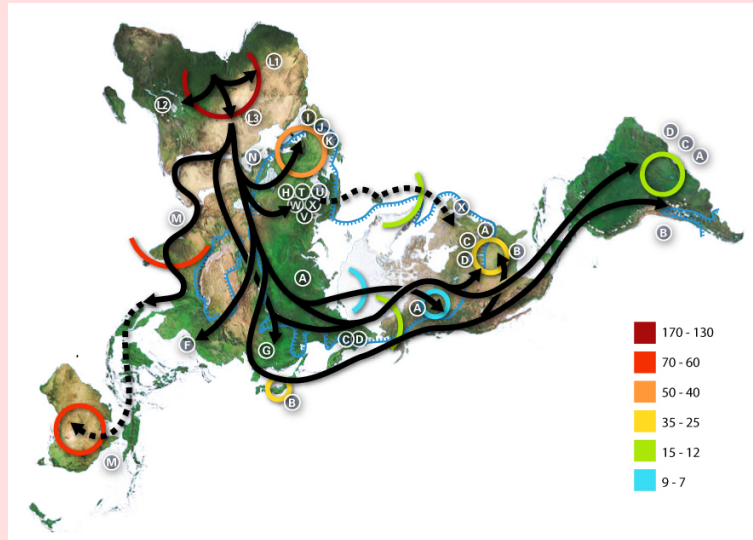
EXAMPLE.

- (a) Hydrothermal ecosystems (John Nelson); Spilhaus World Ocean map.



<https://www.esri.com/arcgis-blog/products/arcgis-pro/mapping/spilhaus-more-like-thrillhaus/>

(b) Prehistoric human migrations via mitochondrial genetics (unknown); Dymaxion map.



https://commons.wikimedia.org/wiki/File:World_map_of_prehistoric_human_migrations.jpg

Mathematics of voting

1. Social choice theory & voting systems

- (a) How do you decide a winner in a contest with three or more options?
- (b) **Plurality voting.** The option with the highest number of votes wins, regardless of what percentage of the voting population voted for that option.
- (c) **Majoritarian voting.** To win an option must secure a majority of votes. This can take the form of instant runoff voting (see below) or multiple rounds of voting. Some U.S. states use the latter, sometimes in a “top 2 runoff”.
- (d) **Approval voting.** Each voter may approve or disapprove of each option. The option with the greatest total approval wins.
- (e) **Rank-order voting** Each voter ranks the options. A predefined *social welfare function* takes all of the individual preferences and returns a list of group preferences.
 - Criteria you might want for a rank-order voting system (i.e., order the options by preference)
 - **Universality** The social welfare function always yields a unique, complete ranking of group preferences.
 - **Pareto efficiency** If every individual ranks *A* above *B* then so does the group.
 - **Independence of irrelevant alternatives (I.I.A.)** The group’s preference between choices *A* and *B* depends only on the individual preferences between *A* and *B*; (from Wikipedia:) “the election outcome remains the same even if an option that cannot win is added”
 - **Nonexistence of dictators** When there are 2 or more voters the group’s preferences are not determined alone by the preferences of any one voter.

THEOREM (Arrow’s Impossibility Theorem). There is no rank-order voting system that satisfies all four of the above conditions.

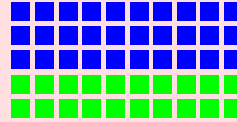
- **Condorcet winner criterion.** The **Condorcet winner criterion** asks that if an option would win in a 2-option vote against every other option, that option must win the election.
- **Other criteria.** There are many more natural-seeming criteria (monotonicity criterion, participation criterion).
- **Instant runoff voting.** If no choice receives a majority of the vote, the lowest-ranked choice is removed and its votes are reallocated to its supporters’ next choices, and that process is repeated until one choice has majority support. This is a form of majoritarian voting. (This method does not satisfy the I.I.A. criterion.)

- **Borda count.** In a vote among n options, each 1st place ranking is worth $n - 1$ points, each 2nd worth $n - 2$ points, and so on. (Borda count satisfies all criteria except for I.I.A.)

2. Gerrymandering

- (a) **Definition** Gerrymandering is the practice of defining electoral boundary districts to give an “unfair” political advantage to one party or the other.

EXAMPLE. A standard example: Consider 50 precincts to be apportioned into 5 contiguous districts of 10 precincts each.



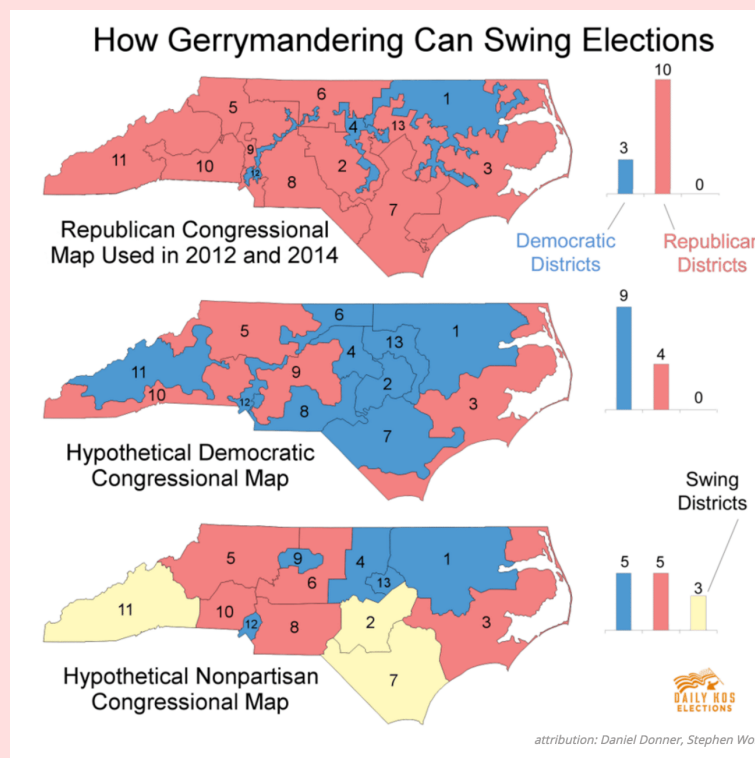
What are ways that we can define district boundaries so that we end up with proportionate representation? Disproportionate?

- (b) **Two main techniques: Packing and cracking.**

EXAMPLE (North Carolina).

In the 2018 election, Democratic candidates received about 51% of the votes for the N.C. Senate and N.C. House of Representatives, but in both cases they won a minority of seats: 42% and 46% of the seats, respectively.

In the 2020 election, Democratic candidates for the U.S. House of Representatives received a plurality of votes (50.0%–49.4%) but only won 5 of the state’s 13 House seats.



<https://www.washingtonpost.com/news/wonk/wp/2016/10/28/how-to-gerrymander-your-way-to-a-huge-election-victory/>

ACTIVITY. Draw a map (not necessarily rectangular) of 81 squares, each of which represents a precinct marked with one of two colors. Exchange with one of your neighbors and attempt to gerrymander the map into 9 contiguous districts (of 9 precincts each) creating as disproportionate of an outcome as you can in favor of one color or the other.