

MATH 110: Class 04

August 22: Numbers Beyond the Integers

For Wednesday, August 24:

Homework Problem Set 1 due

Numbers Beyond the Integers

1. Review of some sets of numbers we've already seen:

- Natural numbers: $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$
- Integers: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- Prime numbers: $\mathbb{P} = \{2, 3, 5, 7, 11, \dots\}$

2. Rational Numbers, \mathbb{Q}

- Recall that $+$, $-$ and \times are all binary operations on \mathbb{Z} .
- If n is divisible by p , that is, if $n = mp$ for integers n, m, p , then we say that $m = n \div p$. If not, then by definition there is no integer m such that $m = n \div p$. This can be inconvenient, as we often care about situations where we want to divide a quantity into pieces smaller than a unit (e.g., splitting one pizza pie between two or more people). So, if we want to make sense of division of integers, we need to expand the set of numbers we're looking at.
- **Definition of rational numbers.** Informally we define the set of **rational numbers**, which we denote by the symbol \mathbb{Q} , to be the set of all ratios of integers: $\frac{a}{b}$, $b \neq 0$.
 - Different choices of a, b can give the same ratios of integers, e.g., $\frac{1}{2} = \frac{2}{4}$ —so we consider $\frac{1}{2}$ and $\frac{2}{4}$ to be the same number. Multiplying both sides of $\frac{a}{b} = \frac{c}{d}$ tells us that those two fractions are equal if and only if $ad = bc$ —a condition that requires only integer multiplication to evaluate. We can always choose a representation $\frac{a}{b}$ such that a and b have no prime factors in common (this is called “reduced form”, and it is unique up to replacing a and b by their negatives).
 - By definition we can think of any integer n as the ratio $\frac{n}{1}$.
 - Why do we have to exclude the possibility that the denominator is 0?
- **Adding and subtracting rational numbers.**

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}.$$

- **Multiplying rational numbers.**

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

- **Dividing rational numbers.**

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

Notice that this formula requires that $\frac{c}{d} \neq 0$, that is, that $c \neq 0$; so, just as with integers, we cannot divide rational numbers by zero.

EXERCISE. If $a = \frac{2}{3}$ and $b = \frac{1}{4}$, compute

- (a) $a + b$
- (b) $a - b$
- (c) $a \times b$
- (d) $a \div b$

- **Inequalities and rational numbers** Multiplying both sides by bd shows that for any $a, b, c, d \geq 0$ we should define $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$, and make appropriate adjustments when some of a, b, c, d are negative. We define $>, \leq, \geq$ analogously.

- **Fields** The fact that we can add, subtract, multiply, and divide rational numbers (except for division by 0), and that those four operations satisfy the usual identities, makes \mathbb{Q} an example of a **field**. We'll see more examples of fields soon.

– Even though \div is not a binary operation on \mathbb{Q} , it *is* a binary operation on $\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$. The analogous statement is true for any field.

- **Absolute value of a rational number**

$$\left| \frac{a}{b} \right| = \frac{|a|}{|b|}.$$

- **Cardinality of the rational numbers.** Using a “zig-zag” pattern we can show that the rational numbers \mathbb{Q} has the same cardinality as \mathbb{N} —in other words, there are just as many natural numbers as there are rational numbers.

3. Real numbers, \mathbb{R}

- Some quantities that we are interested in, like the ratio π of any circle's circumference and its diameter, or Euler's constant, e , are not rational numbers.

EXAMPLE. There is no rational number $\frac{a}{b}$ whose square is 2.

Suppose there were; we might as well assume that a and b have no common factors. So, $2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$. Rearranging gives the equality $a^2 = 2b^2$ of integers. Since 2 is a factor of a^2 it must also be a factor of a , so write $a = 2c$. Substituting gives $2b^2 = a^2 = (2c)^2 = 4c^2$, and dividing both sides by 2 gives $b^2 = 2c^2$. By the same reasoning as before, 2 must also be a factor of b . But that contradicts that a and b have no common factors, so our original assumption (that there is a rational number whose square is 2) is false.

- The idea of the real numbers is to fill in the “gaps” between rational numbers, so that we have an entire line's worth of numbers. We call the resulting set the **real numbers** and denote it by \mathbb{R} ; like \mathbb{Q} , \mathbb{R} is a field.
- An **irrational number** is a real number that is not rational (that is, an element of $\mathbb{R} \setminus \mathbb{Q}$).
- **Decimal representations of real numbers.** It is often convenient to represent real numbers with a *decimal representation*. Recall that the expression $a_k a_{k-1} \cdots a_1 a_0 . a_{-1} \cdots a_{\ell+1} a_\ell$ is our decimal notation for

$$a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \cdots + 10a_1 + a_0 + 10^{-1}a_{-1} + \cdots + 10^{\ell+1}a_{\ell+1} + 10^\ell a_\ell.$$

We define $a_k a_{k-1} \cdots a_1 a_0 . a_{-1} \cdots$ (so, possibly with infinitely many nonzero decimal places) to be the smallest real number at least as large as all of the numbers

$$a_k a_{k-1} \cdots a_1 a_0, \quad a_k a_{k-1} \cdots a_1 a_0 . a_{-1}, \quad a_k a_{k-1} \cdots a_1 a_0 . a_{-1} a_{-2}, \quad \dots$$

– Warning: Some numbers have more than one decimal representation, for example, $0.\overline{9} = 1$; there is no “largest number smaller than 1”.

- A real number is rational if and only if its decimal representation terminates or the digits in its expansion eventually repeat.

If the digits of a number after a decimal place eventually repeat, we often indicate as much by drawing a bar over the repeating digits.

EXAMPLE.

(a) $\frac{1}{6} = 0.16666 \dots = 0.1\overline{6}$

(b) $\frac{1}{7} = 0.142857142857 \dots = 0.\overline{142857}$

EXERCISE. Write the following rational numbers using bar notation.

(a) $\frac{1}{3} = 0.33333 \dots$

(b) $\frac{1}{13} = 0.07692307692 \dots$

- **What is the cardinality of \mathbb{R} ?** Since $\mathbb{N} \subset \mathbb{R}$, \mathbb{R} must have infinitely many elements. Does \mathbb{R} have the same cardinality (\aleph_0) as \mathbb{N} ?

– **Cantor's Diagonalization Argument** Suppose it does. Then there is some one-to-one correspondence between \mathbb{N} and \mathbb{R} ; write out the decimals corresponding to $0, 1, 2, \dots$. The result will look something like the following:

0	\leftrightarrow	0.49082105...
1	\leftrightarrow	0.67957697...
2	\leftrightarrow	0.45557175...
3	\leftrightarrow	0.08868052...
4	\leftrightarrow	0.27144647...
\vdots		\vdots

0.23715...

We'll now use a clever trick to write down a new number that's not in our list. That will establish that not all of the real numbers appear in our list and thus that our assumption that there is a one-to-one correspondence is wrong.

Look at the first number after the decimal place in the number corresponding to 0; in our example it's a 4. Pick any digit other than 0, 4, 9, and we'll choose our new number to have that digit, say, 2, in the tenths place. Then, do the same for the second digit after the decimal place in the number corresponding to 1, which is 7; again we'll choose any digit other than 0, 7, 9 for the hundredths place of our new number, say, 3, and so on. Our new number may be 0.23715.... Now, because of the way we constructed this number, it's not the first number on our list, since the two numbers have different digits in tenths place. Likewise, it differs from the second number in our list, since the two numbers have different digits in the hundredths place, and so on. Thus, our new number differs from all of the numbers in our list, i.e., it's not *on* the list. So, our assumption that there is a one-to-one correspondence is wrong.

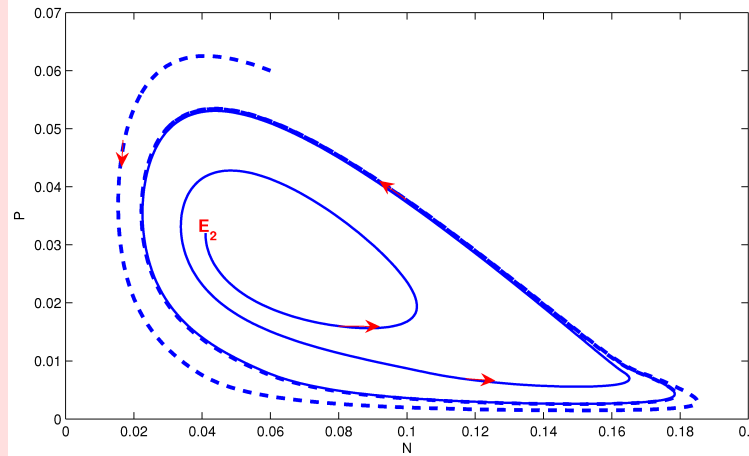
- We thus conclude that even though \mathbb{N} and \mathbb{R} are both infinite, they have different cardinalities; put another way, there is more than one “size” of infinity! The technical term for the cardinality of \mathbb{R} is the *cardinality of the continuum*, and its symbol is \mathfrak{c} .

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Representing two pieces of numerical data at the same time: The Cartesian plane

1. **The Cartesian plane.** Often we are simultaneously interested in two numerical data at the same time—for example, if we are simultaneously studying the populations of two species, or recording both the horizontal and vertical displacement of an object. In many cases it is convenient to use the Cartesian plane, which is a visual representation of the pairs (a, b) of real numbers. We often denote the Cartesian plane by \mathbb{R}^2 , which is a shorthand notation for $\mathbb{R} \times \mathbb{R}$. Typically we label the horizontal axis the x -axis and the vertical the y -axis, so that (a, b) represents the situation wherein $x = a$ and $y = b$.

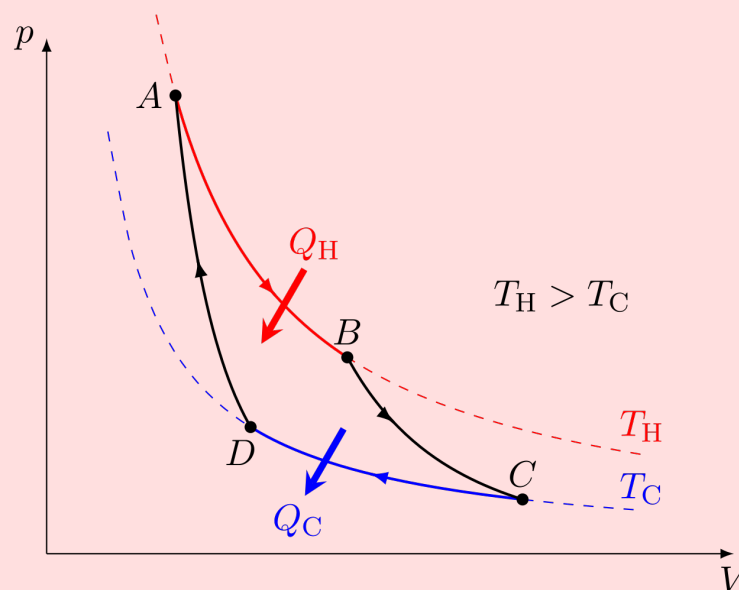
EXAMPLE (Population ecology: Predator–prey model with a stable limit cycle).



Li, B.; Liu, S.; Cui, J.; Li, J. A Simple Predator-Prey Population Model with Rich Dynamics. *Appl. Sci.* (2016) **6** 151. <https://www.mdpi.com/2076-3417/6/5/151>

This plot shows the evolution of a predator–prey model over time. Here, N is (in some units) the population of a prey species and P the population of the predator species. Notice that for both trajectories, corresponding to different starting populations of the two species, approach a stable loop—like the cycles in our (discrete dynamics) icebreaker activity. (Here E_2 marks an *unstable equilibrium*.)

EXAMPLE (Thermodynamics: Carnot cycle).



This plot shows the evolution of the volume (V) and pressure (P) of a fixed quantity of gas in a Carnot engine. The total amount of work done per cycle corresponds to the area of the region enclosed by the trajectory.

EXAMPLE (Cartography: Mercator projection).



This plot shows the Mercator projection—a common map projection—of the earth. In this case, the two dimensions (horizontal and vertical) represent two physical directions of travel (W–E and N–S, respectively). Like any map projection, it distorts some features (in this case, areas, and quite badly near the poles) at the expense of preserving others (angles; constant headings are straight lines). We'll talk more about map projections later.

Numbers Beyond the Integers Redux

1. Complex numbers, \mathbb{C}

- Recall that the square of any real number is positive (or 0), so there is no real number x that satisfies $x^2 = -1$. Adding 1 to both sides shows that, equivalently the polynomial $x^2 + 1$ has no real roots.
- The imaginary unit, i .** Just as we expanded \mathbb{Z} to \mathbb{Q} and \mathbb{Q} to \mathbb{R} , we can expand \mathbb{R} by introducing a new number that satisfies $i^2 = -1$ (that is, is a root of $x^2 + 1$).
- We'd also like the addition and multiplication operations to be defined on our new, larger set, so we must also include the numbers bi , where b is real—these are called **imaginary numbers**—and combinations of real and imaginary numbers. Do we need to add in any more quantities for addition, subtraction, and multiplication to be defined on our set?
- Addition and subtraction.** Provided that we want addition and multiplication on our new set to obey usual rules (in particular, commutativity, associativity, and distributivity of \times over $+$), we must have

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i,$$

and this sum has the form $A + Bi$.

- Multiplication.** We must have $(a + bi)(c + di) = ac + adi + bci + bdi^2$. By definition $i^2 = -1$, so we have

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i,$$

and again this product has the form $A + Bi$.

- So, we define the **complex numbers**, \mathbb{C} , to be the set of all quantities of the form $a + bi$, and \mathbb{C} comes naturally with operations $+$, $-$, \times .

- **The complex numbers and the Cartesian plane.** We can think of the complex numbers geometrically as the Cartesian plane by identifying numbers $a + bi$ respectively with coordinate pairs (a, b) .
- **Why do we care about complex numbers?**
- **The norm of a complex number.** The **norm** of a complex number z is the distance $|z|$ from the origin to z ; by the Pythagorean Theorem, $|a + bi| = \sqrt{a^2 + b^2}$. Notice that $|a + bi|^2 = a^2 + b^2 = (a + bi)(a - bi)$.
- **Complex conjugation** The identity in the previous item suggests that there is a special relationship between each number $a + bi$ and $a - bi$. We say that these two numbers are **complex conjugates** of one another. We denote the complex conjugate of z by \bar{z} . In this notation the above identity is $|z|^2 = z\bar{z}$.
- **Division of complex numbers.** Motivated by the above identity, we can divide as follows:

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$

Notice that the expression on the right is valid unless $c^2 + d^2 = 0$, that is, $c + di = 0$. In part because of this \mathbb{C} is also a field.

EXAMPLE. If $z = 2 + i$ and $w = 1 + 2i$, compute

- $z + w$
- $z - w$
- $z \times w$
- $z \div w$

EXERCISE (The geometric meaning of complex multiplication)

Pick any three noncollinear complex numbers z_1, z_2, z_3 (noncollinear means that they aren't all sitting on the same line), ideally with simple coefficients, and draw the triangle with those points as vertices. Compute the following triples of numbers and plot the triangles with them as vertices.

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$$-z_1, \quad -z_2, \quad -z_3.$$

What effect does multiplying by -1 have on the triangle?

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$$\bar{z}_1, \quad \bar{z}_2, \quad \bar{z}_3.$$

What effect does applying the complex conjugation map have on the triangle?

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$$2z_1, \quad 2z_2, \quad 2z_3.$$

What effect does multiplying by 2 have on the triangle?

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$$iz_1, \quad iz_2, \quad iz_3.$$

What effect does multiplying by i have on the triangle?

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$$(1 + i)z_1, \quad (1 + i)z_2, \quad (1 + i)z_3.$$

What effect does multiplying by $1 + i$ have on the triangle?