

MATH 110: Class 03

August 19: Number sequences and pattern-finding (continued); Counting; Number theory

For Monday, August 22 (Numbers beyond the integers, Probability):

Reading Five Paradoxes With Probabilities That Will Puzzle You, Marcel Moosbrugger, *Towards Data Science*, 2021 May 8.

Project Project check-in

Sequences and pattern-finding (continued)

1. The Fibonacci sequence.

- In the 2nd or 3rd Century B.C.E. the Sanskrit poet-mathematician Pingala was interested in (and found a rule for) enumerating all the different patterns of a given length built from short syllables (length 1 unit) and long syllables (length 2 units).

length	count	patterns
1	1	■
2	2	■ ■, ■■
3	3	■ ■ ■, ■ ■■, ■■■
4	5	■ ■ ■ ■, ■ ■ ■■, ■ ■■■■, ■ ■ ■ ■■, ■ ■ ■■■■
⋮	⋮	⋮

- What number comes next?
- How do we find a general pattern?
- **Definition.** Define

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \text{ for } n > 1.$$

The sequence starts thusly:

$$0, 1, 1, 2, 3, 5, \dots$$

- Named for Leonardo of Pisa, who described the sequence in 1202.
- Fibonacci numbers occur frequently in applications (computer science) and nature, e.g., the arrangement of leaves on a stem or the flowering of an artichoke.
- Tessellating squares diagram; applications to the Golden Ratio ϕ /Classical aesthetics. What happens to the ratios F_{n+1}/F_n as n gets large?
- Counting the number of genetic contributors to a typical biological male's X chromosome in each previous generation.
- Tiling a $2 \times n$ board with dominos.

Counting

There's much more to say concerning counting, but we'll focus on three topics: The factorial; combinations and permutations; and the Pigeonhole Principle.

1. Factorials.

- How many ways can we arrange three objects in a row? Four objects? All of the students in this class?

- The **factorial** of a positive integer n is the product of n and all of the positive integers smaller than it:

$$n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1.$$

For good reason (why?) we declare that the factorial $0!$ of 0 is 1.

EXERCISE (Factorials of small numbers) Compute the values of the factorials $0!, 1!, 2!, \dots, 8!, 9!$.

2. Permutations and combinations.

- How many ways can we choose a class president, class vice president, and class secretary?
- The number of k -permutations of n objects—choices of k things from n distinct objects, counting order—is denoted ${}_nP_k$ (or one of many similar notations), and for $n \geq k \geq 0$,

$${}_nP_k = \frac{n!}{(n-k)!}.$$

- How many ways can we choose a class council of 3 students?
- The number of k -combinations of n objects—choices of k things from n distinct objects, not counting order—is denoted ${}_nC_k$ (or one of many similar notations) or $\binom{n}{k}$, and for $n \geq k \geq 0$,

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

3. The Pigeonhole Principle

EXAMPLE. (The sock problem). Suppose you are pulling socks out of a drawer in the dark, so that you can't see their colors. If you have two colors of socks, say, black and white, how many socks do you need to pull out to guarantee that you have at least one matching pair? What about two matching pairs? What if you have three colors of socks? Seven colors?

EXERCISE. How many people do you need to have in a room to be guaranteed that at least two people share a birthday (according to the Gregorian calendar)?

THEOREM (The Pigeonhole Principle). If n objects (pigeons) are distributed over $m < n$ places (holes), then at least one hole contains more than one object.

EXAMPLE. Show that any among six distinct one-digit numbers, $1, \dots, 9$, there is at least one pair of numbers whose sum is 10. Is the same true if you instead take only five distinct one-digit numbers?

EXERCISE.

- Use the Pigeonhole Principle to show that there are two students in our class who were born in the same calendar month (not necessarily in the same year). Can we use the Pigeonhole Principle to show that there are two students in our class who were born on the same day of the month (not necessarily in the same month or year)?
- Supposing that all 1525 students at Guilford College have a height between 1 m (about 3'3") and 2.25 m (about 7'5"), show that two students at the college have the same height to the nearest millimeter.
- Suppose you have n people in a room. Show that there are at least two people in the room who have the same number of friends in the room. (Here we assume that friendship is "symmetric", that is, if Person A is a friend of person B, then person B is also a friend of person A.)

Number theory

- Classical number theory is essentially the study of the integers.
- Divisibility.** If m and n are integers, we say that m **divides** n (or that n is **divisible** by m) if there is an integer p such that $n = mp$. (Note that this implies that p also divides n .) If m and n are both positive, then we say that m is a **factor** of n .

EXAMPLE.

The factors of 12 are 1, 2, 3, 4, 6, 12.

The factors of 5 are 1, 5.

EXERCISE.

Find all of the factors of the following numbers:

- (a) 4
- (b) 6
- (c) 7
- (d) 15

3. Prime numbers.

- If a number n has a factor other than 1 and n , by definition we can write n as a product of two smaller natural numbers, and we say that n is **composite**.
- On the other hand, if $p \neq 1$ cannot be written as a product of two smaller natural numbers, we say that p is **prime**.

EXAMPLE. The first few prime numbers are 2, 3, 5, 7, 11, ...

- Since any composite number can be written as a product of smaller numbers, *every* positive integer is a product of prime numbers. Proof (sketch): Keep factoring factors into products of smaller numbers until you can't factor any of the factors any more.)

EXAMPLE. Use a tree diagram to write 24 as a product of prime numbers.

EXERCISE. Use a tree diagram to write each of the following numbers as a product of prime numbers:

- (a) 28
- (b) 60

- We observed in the previous exercise that even when we factored the given numbers in different ways, we always ended up with the same prime factorization. A short extension of the above argument shows that this is always true.

Theorem. Every positive integer can be factored uniquely as a product of primes.

- Even though prime numbers have been studied for thousands of years, we still have many open questions about them. Proving (or disproving) the Riemann Hypothesis, which is widely considered one of the most important open problems in mathematics today, can be regarded as a statement about how the prime numbers are distributed.

EXERCISE. In small groups, determine all of the prime numbers smaller than 100.

- **The Sieve of Eratosthenes.** One can systematically produce a list of the prime numbers by writing the list 1, 2, 3, ... of positive integers, then crossing out the multiples of 2 larger than 2 (4, 6, 8, 10, 12, ...), then crossing out the multiples of 3 larger than 3 (we may ignore numbers that are already crossed out, so we cross out (9, 15, 21, 27, 33, ...)). Since 4 is crossed out, we know that every multiple of 4 is already composite (indeed, we cross those multiples out when we crossed out the even numbers larger than 2). Then, we cross out the multiples of 5 larger than 5 that we haven't crossed out yet: (25, 35, 55, 65, 85, ...).
- **An algorithm for checking whether a number is prime.** To check whether n is prime, check whether each prime number no larger than \sqrt{n} is a factor of n . If any are factors, n is not prime. Otherwise n must be prime.

EXERCISE. Use the given algorithm to determine whether the following numbers are prime:

- (a) 127
- (b) 143

THEOREM (Euclid's Theorem) There are infinitely many prime numbers.

– Proof: A clever trick!

- **Mersenne primes.** A **Mersenne prime** is a prime that is one less than a power of 2, that is, a prime of the form $2^n - 1$. It turns out that $2^n - 1$ is always composite if n is, so Mersenne primes all have the form $2^p - 1$ for some prime number p .

Find the four smallest Mersenne primes.

Remark: The fifth-smallest Mersenne prime is already somewhat large: 8191.

- The largest number known to be a Mersenne prime known is $2^{82\,589\,933} - 1$ —this number was only discovered to be prime in December 2018. It has over 24 million digits, and is, in fact, the largest known prime number. We only know 51 Mersenne primes so far, and we don't know whether there are infinitely many.

- **Twin primes.** Two prime numbers are called **twin primes** if they differ by 2: The smallest pairs of twin primes are (3, 5), (5, 7), and (11, 13)

Find the next three smallest pairs of twin primes.

- We don't know whether there are infinitely many pairs of twin primes. We call the conjecture that there are the Twin Prime Conjecture.

4. **Perfect numbers.** We call a number n perfect if the sum of its proper divisors—that is, the divisors other than n itself—is equal to n itself.

The smallest perfect number is 6: The factors of 6 are 1, 2, 3, and 6 itself, and $1 + 2 + 3 = 6$.

EXERCISE (The second-smallest perfect number) Find the second-smallest perfect number.

- There is a surprising relationship between Mersenne primes and perfect numbers—this is part of why people became interested in studying Mersenne primes.

THEOREM (Euclid–Euler Theorem). If $2^p - 1$ is a Mersenne prime, then $2^{p-1}(2^p - 1)$ is perfect, and all even perfect numbers can be produced this way.

EXERCISE (The third-smallest perfect number). Use your list of the smallest Mersenne primes from an earlier exercise together with the Euclid–Euler Theorem to find the third-smallest even perfect number. (Answer: 496.)

- We don't know whether there are any odd perfect numbers, but if there are any, they must be *very* large: We know that any odd perfect number must be larger than $10^{1\,500}$.