

MATH 110: Class 07

August 25: Axiomatic reasoning, Euclidean geometry in 2 dimensions

Much of today's material is adapted from *A Mathematics Sampler*, Berlinghoff et al., §3.

Axiomatic reasoning

1. Axioms are the ground rules under which we do mathematics—statements that accept as true or self-evident.
2. **Axiomatic systems.** An axiomatic system is a framework of rules and statements in which we deduce theorems.
 - An axiomatic system entails four types of data:
 - **Defined terms.** Terms specified by a *characteristic property*, which we use to determine unambiguously whether or not an object meets the definition.
 - * The statement that “a square is a polygon with 4 sides” is true, but it is not a definition of a square, as there are (many) 4-sided polygons that are not squares. A definition includes a full description of the characteristic property; a good definition for a square might be: “a **square** is a polygon with 4 sides whose angles are right angles and whose sides all have the same length.”
 - * A set of definitions should not be circular: It's true that both
 - a composite number is a natural number greater than 1 that is not prime, and that
 - a prime number is a natural number greater than 1 that is not composite,but taken together these definitions are *circular*: They depend on one another, and with them one cannot actually decide whether a natural number greater than 1 is prime or composite.
 - **Undefined terms.** The prohibition on circular definitions means that our axiomatic system must have some *undefined terms*—they are not defined, but nor are they meaningless: The terms inherit meaning from how they are used, i.e., from other data in the axiomatic system.
 - **Axioms.** Axioms are the mathematical statements—stated in terms of defined and undefined terms—that we accept as true for our system.
 - **Theorems.** Theorems are true statements that we deduce from axioms.

EXAMPLE (Projective geometry). In a painting that faithfully represents perspective, the physical lines that depict parallel lines stretching into the distance physically meet on the image. So, visual artists are sometimes practically interested in a geometry wherein any two lines intersect (even if only “at infinity”).

The axioms of projective geometry are:

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- P1. There exist at least one point and one line.
 - P2. Through any two distinct points there is exactly one line.
 - P3. Any two distinct lines intersect in exactly one point.
 - P4. Any line contains at least three points.
 - P5. Not all points lie on the same line.
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Theorems deducible from these axioms include:

Theorem. If there are only finitely many points, then there is a natural number $N > 1$ such that:

- There are exactly $N^2 + N + 1$ points.
- There are exactly $N^2 + N + 1$ lines.
- Each line contains exactly $N + 1$ points.
- Each point lies on exactly $N + 1$ lines.

Corollary. There are at least 7 points and 7 lines.

- **Material and formal axiomatic systems and models.**

- **Material axiomatic systems.** A **material axiomatic system** is motivated by reality, and in particular its undefined terms have meanings derived therefrom.
- **Formal axiomatic systems.** In a **formal axiomatic system** undefined terms are placeholders than need not have any real-world analogue—they essentially behave like algebraic symbols that can be manipulated. A formal axiomatic system specifies its own logical rules.

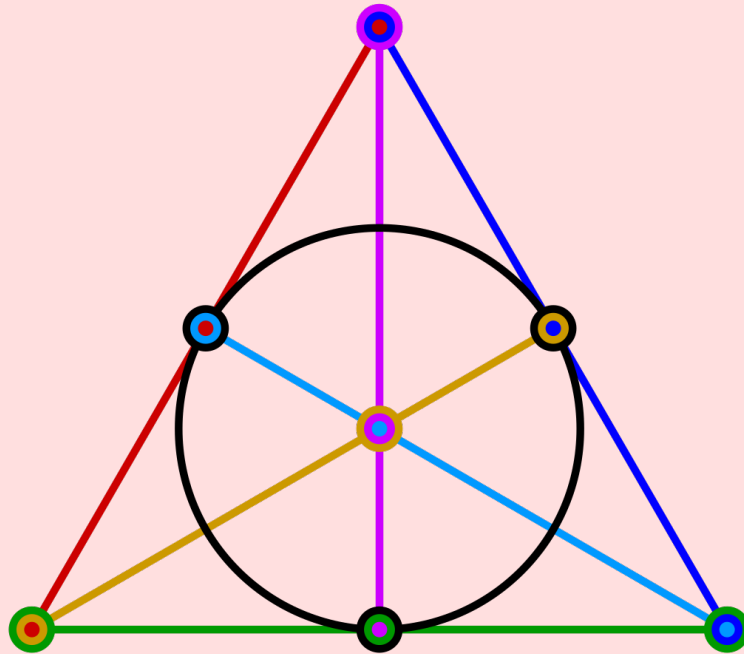
EXAMPLE (**Projective geometry**). In our previous example the use of the undefined terms “point” and “line” appeal to real-world notions we already have, making this a material axiomatic system. If replaced those terms respectively with the symbols $*$ and \sim , and the descriptive verbs are replaced with less descriptive ones, then the resulting system would be more emphatically formal.

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- P1. There exist at least one $*$ and one \sim .
 - P2. If A and B are distinct $*$ then there is exactly one \sim that is related to both.
 - P3. If A and B are distinct \sim then there is exactly one $*$ that is related to both.
 - P4. There are at least three $*$ s related to each \sim .
 - P5. For any \sim there is an $*$ not related to the \sim .
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- **Interpretations and models.** An **interpretation** of an axiomatic system is an assignment of specific meanings to undefined terms. If an axiom becomes a true statement when interpreted thus, we say that the interpretation **satisfies** the axiom. A **model** of an axiomatic system is an interpretation that satisfies all of the axioms of the system.

EXAMPLE.

- (a) (**The Fano plane**). Our earlier corollary shows that this is the smallest model of projective geometry: No model contains fewer than 7 points.



- (b) (**Real projective plane**). Declare $*$ to be “line through the origin in \mathbb{R}^3 ” and \sim to be “plane through the origin”. A line ℓ is related to a plane Π if $\ell \subset \Pi$, that is, if the plane contains the line.

- Any statement that can be proved from the axioms of an axiomatic system is true of every model of the system. This lets us deduce facts about different mathematical objects with a single proof.
- Later in today’s meeting we’ll see an example of a proof of a mathematical statement from axioms.

- **Consistency**

- The negation of a statement is another statement that is true exactly when the original statement is false; cf. the propositions P and $\neg P$ we discussed in proposition logic.
- **Consistency and inconsistency.** An axiomatic system is **inconsistent** if there a statement such that from the axioms you can prove both the statement and its negation.

EXAMPLE.

Consider a (trivial) axiomatic system with two axioms:

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- N1. $*$ relates to \sim .
 - N2. $*$ does not relate to \sim .
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Since you can prove from the axioms both the statement “ $*$ relates to \sim ” (via N1) and its negation (via N2), this axiomatic system is not consistent.

- **Completeness.** An axiomatic system is **complete** if, for every statement in the language of the axioms, either that statement is provable or its negation is.

Gödel’s Incompleteness Theorem (Paraphrase, ignoring some technical issues:) Any axiomatic system from which you can deduce the basic arithmetic of natural numbers cannot be both consistent and complete. Put another way, for any consistent set of axioms, there will be some statements about the natural numbers that the axiomatic system cannot prove.

- *Remark:* This theorem is one of the most important achievements of 20th Century mathematics.

Euclidean geometry in two dimensions

1. **The axioms of Euclidean geometry.** In Euclid's *Elements*, all of Euclidean geometry is derived from a list of five “common notions” (axioms) and five “postulates” together with 23 definitions. Today we treat the terms “common notions” and “postulates” interchangeably. The assertions he calls postulates are as follows:

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- E1. If A, B are distinct points, there is exactly one line passing through A and B .
 - E2. Any line segment can be extended to a straight line.
 - E3. Given any point O and radius r we can construct the circle of radius r centered at O .
 - E4. All right angles are congruent (i.e., equal).
 - E5. If a line A intersects two other lines B, C , and the sum of the two interior angles on one side of A are smaller than the sum of two right angles, then B and C intersect on that side of A .
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All 465 propositions in the 13 books of Euclid's *Elements* are deduced from the above five axioms.

2. **Straightedge-and-compass constructions** The constructions in axioms (E1) and (E2) are exactly those that can be made with a straightedge, and the construct in axiom (E3) is exactly that which can be made with a compass. Since these are the only constructions mentioned in the axioms, all other geometric constructions in Euclid's axiomatic system must be derived from these. We call such constructions **straightedge-and-compass constructions**.
3. In the proof of any true statement (in any axiomatic system), each step must be justified by one of the axioms (or a statement already justified by axioms).

EXAMPLE (Construction of an equilateral triangle with a given side).

Proposition 1. Given any line segment AB (so, with endpoints A, B), we can construct an equilateral triangle one of whose sides is AB .

Proof.

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|---|---|
| 1. Draw a circle with center A and radius $ AB $. | (E3) |
| 2. Draw a circle with center B and radius $ AB $. | (E3) |
| 3. From an intersection point of the two circles—call it C —construct segments AC and BC . | (E1) |
| 4. Since A is the center of the circle through B and C , $ AB = AC $. | definition of circle |
| 5. Since B is the center of the circle through A and C , $ AB = BC $. | definition of circle |
| 6. By (4) and (5), $ AC = BC $. | equality is transitive * |
| 7. Since all three line segments AB, AC, BC have the same size, $\triangle ABC$ is equilateral. ■ | definition of equilateral triangle |
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* The transitivity of equality—that if P and Q are equal quantities and so are P and R , then Q and R are equal quantities—is Euclid's fourth “common notion”.

EXAMPLE (Sample propositions from Book I of the *Elements*).

Proposition	Statement
4.	(SAS)
8.	(SSS)
9.	How to bisect a given angle.
10.	How to bisect a given line segment.
15.	Opposite angles formed by the intersection of two straight lines have the same measure.
47.	(Pythagorean Theorem)

4. **The Parallel Postulate** Axiom E5 is sometimes called the **Parallel Postulate**, and Euclid treats it differently from the rest: He doesn't use it in *Elements* until Proposition 27.

- **Playfair's Axiom** It is easier to see the philosophical relevance of the Parallel Postulate if we replace it with another axiom, E5', called **Playfair's Axiom**, which has the property that the axiomatic system {E1, E2, E3, E4, E5} is equivalent to (has the same theorems as) {E1, E2, E3, E4, E5'}.

E5'. Given a line ℓ and a point p not on ℓ there is exactly one line that passes through p and is parallel to ℓ .

- What happens if we assume axioms E1, ..., E4 but not E5 or E5'? Can we find a model that fits axioms E1–E4 that has different theorems than Euclidean geometry does? Answer: Yes! Examples?

5. **Basic shapes in the Euclidean plane**

- Circles, triangles, polygons, regular polygons
- Which regular polygons are constructible using a straightedge and compass? (The answer is, maybe surprisingly, related to special prime numbers called *Fermat primes*!)