

MATH 110: Class 10

August 30: Symmetries and groups

Symmetry

1. What is symmetry?

- **Why do we care?**
- Roughly speaking a symmetry is a transformation of an object that preserves a feature of interest of that object.
- **The identity symmetry.** Note that by the above informal definition, the transformation that leaves an object alone is always a symmetry—we call it the **identity (symmetry)** and often denote it 1.

2. Symmetries of some simple geometric shapes

EXAMPLE (Symmetries of a (non-square) rectangle). I.e. all possible ways to flip a mattress. A nonsquare rectangle has four symmetries: The identity, reflection across the long axis, reflection across the short axis, and rotation by $\frac{1}{2}$ of a turn.

EXAMPLE (Symmetries of triangles).

- (a) **(Isosceles)** A (non-equilateral) isosceles triangle has exactly two symmetries: The identity (i.e., do-nothing) symmetry, and reflection r across the line bisecting its unique angle.
- (b) **(Scalene)** A scalene triangle has only one symmetry: The identity. In some sense this case is generic: Most shapes have no symmetries other than the identity. But many of the objects that we study have more symmetries—in part because symmetries make objects interesting!
- (c) **(Equilateral)** This is the most interesting case, i.e., the case with the most symmetry. There are three rotations—including the identity, 1, which we can think of as a rotation by a zero angle—as well as reflections across the three angle bisectors.

3. Operations on the set of symmetries.

(a) Composing symmetries to make new symmetries

- **The composition operation on the set of symmetries.** If g, h are both symmetries of an object, then so is the transformation of the object given by applying h first and then applying g ; we denote this new symmetry by $g * h$. Put another way, if we denote by G (“group”) the set of symmetries of an object, there is a natural binary operation

$$* : G \times G \rightarrow G$$

given by declaring $g * h$ to be the symmetry of the object given by applying h and then g ; we can think of this combination the **product** of g and h . For brevity we often write this product using juxtaposition gh , and we write a product of a symmetry g with itself as $g * g = g^2$.

- **Composing with the identity symmetry.** By definition, for any symmetry g in the set of symmetries,

$$1 * g = g * 1 = g.$$

EXAMPLE (Symmetries of a [non-square] rectangle).

Let ℓ denote the reflection across the long axis and s the reflection across the short axis. We can write the rotation by a half-turn as either ℓs or $s\ell$. So, the set of symmetries is

$$\{1, \ell, s, \ell s\}.$$

EXAMPLE (**Symmetries of an equilateral triangle**).

Let s denote the anticlockwise rotation by $\frac{1}{3}$ of a turn, and let r denote one of the reflections across an axis of symmetry.

- How can we write all six symmetries as products of r and s ?
- Is there more than one way of writing a given symmetry?

- **Multiplication table.** As with any binary operation, we can construct a *multiplication table* that specifies the operation.

EXAMPLE ([Non-square] rectangle).

*	1	ℓ	s	ℓs
1	1	ℓ	s	ℓs
ℓ	ℓ	1	ℓs	s
s	s	ℓs	1	ℓ
ℓs	ℓs	s	ℓ	1

ACTIVITY (**Equilateral triangle**). Complete the multiplication table.

*	1	s	s^2	r	rs	rs^2
1	1	s	s^2	r	rs	rs^2
s	s	s^2	1	rs^2	r	rs
s^2						
r						
rs						
rs^2						

(b) **Inverses of symmetries.**

- **Inverses.** Given any symmetry g in the set G of symmetries of some object, by definition the transformation that reverses what g does is also a transformation; we denote it by g^{-1} , and by definition it is characterized by the identities

$$gg^{-1} = g^{-1}g = 1.$$

Remark: We can think of \cdot^{-1} as a unary operator on G .

- **Inverse of a product.** Since

$$gh * h^{-1}g^{-1} = g * 1 * g^{-1} = g * g^{-1} = 1,$$

we have the identity

$$(gh)^{-1} = h^{-1}g^{-1}.$$

- **Inverse of a power.** Specializing the previous formula to the case $h = g$ gives that $(g^2)^{-1} = (g^{-1})^2$; we also denote this quantity by g^{-2} .

EXERCISE. Use the above multiplication table in the previous example to verify that the identity in the previous item holds for the products $\sigma\tau$ and $\sigma^2\tau$.

Groups

1. • **Definition.** We generalize the properties of symmetries that we saw above. A **group** is a set G equipped with a bilinear operation $*$ (*[group] multiplication*) satisfying the following:

- **(Identity)** There is an element $1 \in G$ such that

$$1 * g = g * 1 = 1.$$

- **(Inverses)** For every $g \in G$ there is an *inverse* element, i.e., one satisfying

$$g * g^{-1} = g^{-1} * g = 1.$$

- **(Associativity)** The operation $*$ is associative, that is, for every $g, h, k \in G$,

$$(g * h) * k = g * (h * k).$$

We denote the group by $(G, *)$ or just G , if $*$ is implicitly defined by context.

Remark: In some of our earlier computations we used implicitly the associativity of the operations $*$ from our examples. That use is justified by the fact that we can view symmetries of geometric objects as functions $X \rightarrow X$ for some set X , and we've seen that function composition is associative.

- **Order (size) of a group** If the set G is finite, we say that the **order** of the group is just the cardinality of G ; otherwise, we say that G has infinite order.
- **Order of an element** The **order** of an element g is the smallest positive integer n such that $g^n = 1$.
- **KEY IDEA:**

Just as a number is an abstract representation of a quantity, a group is an abstract representation of the type of symmetry an object has.

EXAMPLE.

- (a) **(Finite cyclic groups)** A **cyclic** group is a group that consists of all powers of a single element g . If g has order n we call the group C_n , and by construction it has n elements. The set of rotations of a regular n -gon, $n \geq 3$ is a cyclic group of order n .
- (b) **(Dihedral groups)** The set of symmetries (both rotations and reflections) of a regular n -gon is the **dihedral group** of order $2n$, which we denote D_{2n} . We can write any element of D_{2n} as a product of the anticlockwise rotation r by $\frac{1}{n}$ turn and any reflection r .
- (c) **(Symmetric groups)** The **symmetric group** S_n of all permutations (shuffles) of a deck of n cards is a group of order $n!$. We sometimes write an element of a symmetric group as a product of *cycles*: The cycle (123) , for example, takes the first card and moves it to the second position, the second to the third position, and the third to the first position (top of the deck).
- (d) **(Alternating groups)** Every shuffle in S_n can be built out of transpositions (permutations exchanging just 2 cards). Half of the shuffles—called *even permutations*—can be built out of an even number of permutations; these comprise the **alternating group** A_n . We have $|A_1| = 1$ and $|A_n| = \frac{1}{2}n!$ for $n > 1$.

- **Abelian groups.** A group $(G, *)$ is **abelian** if $*$ is commutative.

ACTIVITY.

Which of the groups that we've seen are abelian, and which are not?

- **Group isomorphisms.** Sometimes the same group appears in more than one guise. More precisely, two groups $(G, *)$, (H, \star) are isomorphic if there is a bijection $f : G \rightarrow H$ such that $f(g * g') = f(g) \star f(g')$ for all $g, g' \in G$, or, roughly speaking, if G and H have the same multiplication table. We denote an isomorphism between groups G, H by $G \cong H$.

EXAMPLE (**Integer addition modulo n**). The operation $+_n$ on the set $\mathbb{Z}/\langle n \rangle = \{\bar{0}, \dots, \overline{n-1}\}$ defined by

$$\bar{a} +_n \bar{b} = \overline{\text{remainder of } a + b \text{ after dividing by } n},$$

sometimes called clock addition, is a group with identity $\bar{0}$. It is isomorphic to the cyclic group generated by an element g of order n , via the identification $1 \leftrightarrow g$.

EXAMPLE (**Equilateral triangles**).

The group D_6 of symmetries of an equilateral triangle is isomorphic to the symmetric group S_3 (the set of permutations [shuffles] of 3 cards). In plain English, why is that true?

D_6	S_3
1	1
r	(123)
r^2	(132)
s	(12)
rs	(13)
r^2s	(23)

One isomorphism from G to S_3 .

- **Subgroups.** If $H \subseteq G$ and for every $h, h' \in H$ we also have $hh' \in H$, then we say that H is a **subgroup** of G .

EXAMPLE (**The group of rotations of an equilateral triangle**). The composition of any two rotations is another rotation—put another way, any product of two elements of $H = \{1, r, r^2\}$ is another element of that set—so H is a subgroup of G .

- **Cartesian products of groups.** We can combine two groups $(G, *)$, (H, \star) into a new group: The underlying set is just the Cartesian product $G \times H$, that is, the set of pairs (g, h) , where $g \in G$ and $h \in H$, and the group operation \diamond on $G \times H$ multiplies elements of the two factors separately:

$$(g, h) \diamond (g', h') = (g * g', h * h').$$

EXAMPLE ([**Non-square**] **rectangle**) The group G of symmetries of a (non-square) rectangle is isomorphic to $C_2 \times C_2$; the latter is called the **Klein 4-group** and is occasionally denoted K . Denote the elements of C_2 by $\bar{0}$ (the identity) and $\bar{1}$.

G	$C_2 \times C_2$
1	$(\bar{0}, \bar{0})$
ℓ	$(\bar{1}, \bar{0})$
s	$(\bar{0}, \bar{1})$
ℓs	$(\bar{1}, \bar{1})$

One isomorphism from G to $C_2 \times C_2$.

EXERCISE (**The cyclic groups of order 4, 6**).

- Is it true that $C_4 \cong C_2 \times C_2$? Why or why not?
- Is it true that $C_6 \cong C_2 \times C_3$? Why or why not?

- **An interesting example**

Exercise (**The quaternion group, Q_8**).

The quaternion group Q_8 is the group of order 8 with elements

$$\pm 1, \pm i, \pm j, \pm k,$$

where -1 satisfies the identities suggested by the notation $((-1)^2 = 1, (-1)i = i(-1) = -i$, and the consequences of those identities.), $i^2 = j^2 = k^2 = -1$, and $ij = k$.

- Is Q_8 abelian?
- Construct the multiplication table for Q_8 .

- With this last example in hand, together with the direct product, we can construct every group of order < 16 (up to isomorphism) except one, the *dicyclic group* Dic_{12} of order 12).

- **Infinite groups**

- **Definition.** A group $(G, *)$ is **infinite** if $|G|$ is not finite.

- **EXAMPLE.**

- (a) $(\mathbb{R}, +)$

- (b) (\mathbb{R}_+, \times) (this group is isomorphic to $(\mathbb{R}, +)$; can you find an isomorphism between the two?)

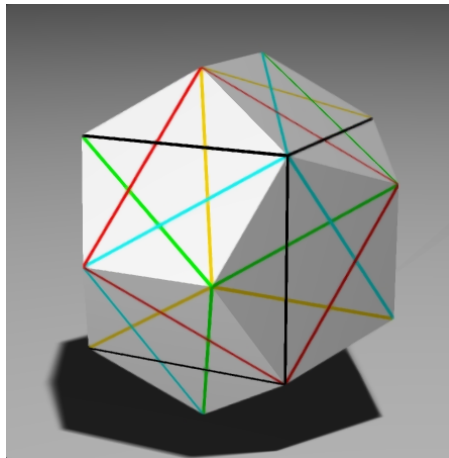
- (c) $SO(2) = \{\text{rotations of a circle}\}$

- (d) $(\mathbb{C} \setminus \{0\}, \times)$ (this group is isomorphic to $SO(2) \times \mathbb{R}_+$)

- (e) $SO(3) = \{\text{rotations of a sphere}\}$ ($SO(3)$ is the only nonabelian group among these examples)

- **Symmetry groups of Platonic solids**

solid	symmetry group	
	rotations only	reflections, too
tetrahedron	A_4	S_4
cube, octahedron	S_4	$S_4 \times C_2$
dodecahedron, icosahedron	A_5	$A_5 \times C_2$



Five cubes inscribed in a dodecahedron, illustrating that the symmetry group of the dodecahedron and icosahedron are isomorphic to A_5 .