

# MATH 110: Class 13

## September 2: Music

### Overview

1. Why would we think that mathematics and music are related?
2. How are they related?

### Pitch

#### 1. Physics

- **Sound** is a matter wave traveling through a medium—usually air.
- We perceive the frequency of a wave—the number of complete cycles per unit time—as the *pitch* of the wave. We define the unit Hertz (Hz) to be one cycle per unit second.
- Humans can hear sound in the range of  $\sim 20$  Hz–  $\sim 20\,000$  Hz (but this varies with person and in particular with age).

EXAMPLE.

frequency (Hz)	note	description
23.12	F <sub>0</sub> <sup>#</sup>	Approximately bottom of range of a dwarf cassowary song
32.70	C <sub>1</sub>	Lowest C on a standard (88-key) piano
65.41	C <sub>2</sub>	Approximate bottom of range of a typical deep bass male voice
87.31	F <sub>2</sub>	Lowest note sung by Freddy Mercury in studio ( <i>Somebody to Love</i> )
116.54	A <sub>2</sub> <sup>#</sup>	Typical (fundamental) frequency of an adult male voice
207.65	G <sub>3</sub> <sup>#</sup>	Typical (fundamental) frequency of an adult female voice
261.63	C <sub>4</sub>	Middle C
440.00	A <sub>5</sub>	Concert A; also denoted A440
1318.51	E <sub>6</sub>	Highest note sung by Freddy Mercury in studio ( <i>It's Late</i> )
4186.01	C <sub>8</sub>	Highest C on a standard piano
9956.03	D <sub>9</sub> <sup>#</sup>	Approximate top of range of a blackpoll warbler song
16744.04	C <sub>10</sub>	Approximate pitch of a CRT display (e.g., an old, pre-flat-panel television)

- The **wavelength** of a sound is the physical length of one cycle (i.e., the distance between successive peaks). The frequency and wavelength of a wave are related by

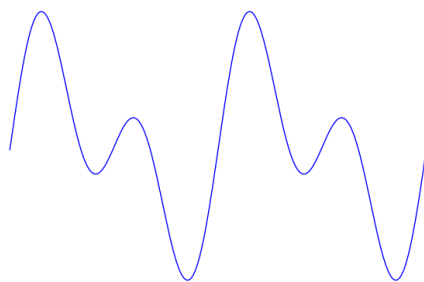
$$\text{speed} = \text{frequency} \times \text{wavelength}.$$

Under typical conditions (dry air at 20°C) the speed of sound is  $343 \text{ m} \cdot \text{s}^{-1}$ .

EXERCISE. Compute the wavelengths of concert A and two other frequencies of your choice from the table in the previous example.

#### 2. Octaves and dividing them

- **Octaves.** A note (a tone of a given frequency) is said to be one *octave* higher than another if the frequency of the first is exactly double that of the second. Pairing the higher note with the lower note is called *doubling*, and the combination of sounds is aesthetically pleasing—why? (Hint: Look at the shape of the physical waves.)



Doubling

- Different musical systems partition an octave into different numbers of notes.
  - Worldwide the most common is a *pentatonic* system, with five primary notes in each octave, and is used, e.g., in some East Asian music and Eastern European folk music.
  - Western music uses a *heptatonic* system, with seven primary tones, and each octave is divided into 12 *half-steps*; traditional Hindustani music uses a similar division.
  - Carnatic (Southern Indian) classical music uses a 22-note scale; the microtones are called *shrutis*.
- **Temperament.** We can divide up octaves into notes in many different ways, called *temperaments*. For simplicity we'll work today only with equal temperament, which means that the ratio of any two successive pitches (steps) is the same as any other.
  - By definition if we have  $n$  equal steps in an octave and  $r$  is the ratio of pitches of two successive notes, we must have  $r^n = 2$ , that is,  $r = \sqrt[n]{2}$ .
- **The perfect fifth and other special intervals.** Just as we find 2 : 1 ratios of frequencies aesthetically appealing, so we also find pairs of notes whose ratios are again simple fractions. The simplest fraction between 1 : 1 and 2 : 1 is 3 : 2—we call this interval (that is, ratio of frequencies) a *perfect fifth*. Other simple fractions include 4 : 3 (a *perfect fourth*) and 5 : 4 (a *major third*). (These concepts are partly attributed to Pythagoras.)



Perfect fifth



Perfect fourth

- **Realizing the perfect fifth.** In particular, a useful feature of an equally tempered scale would be for two notes in the scale to have a ratio at least close to 3 : 2. To say that an interval of  $m$  steps in an equally tempered scale with  $n$  notes is a perfect fifth is to say that

$$2^{m/n} = \frac{3}{2}.$$

Taking the base-2 logarithm of both sides gives the equivalent equation

$$\frac{m}{n} = \log_2 \left( \frac{3}{2} \right) = \log_2 3 - 1 = 0.58496 \dots$$

- **The irrationality of  $\log_2 3$ .** It turns out that the number  $\log_2 3$  is irrational, so there are no integers  $m, n$  such that a perfect fifth is equal to an integer number of steps. Put another way, with respect to representing a perfect fifth (as well as many other intervals), there is no perfect equally tempered scale, i.e., no ideal numbers of steps  $n$  in which to divide an octave.
- **Continued fractions and rational approximations.** The best we can do is find  $m, n$  such that one of the notes in the our equally tempered scale at least approximates a perfect fifth well. To find good rational approximations, we expand  $\log_2 3 - 1$  in a *continued fraction*:

$$\log_2 3 - 1 = \frac{1}{1 + \frac{1}{1 + 2 + \frac{1}{2 + \frac{1}{3 + \dots}}}},$$

which for legibility we sometimes write as

$$\log_2 3 - 1 = [0; 1, 1, 2, 2, 3, 1, 5, 2, \dots].$$

Truncating the continued fraction after each successive steps gives a sequence of increasingly good rational approximations  $\frac{m}{n}$  to  $\log_2 3 - 1$ :

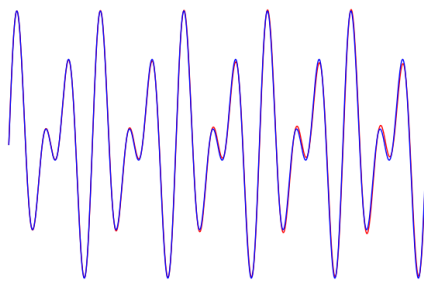
$$1, \frac{1}{2}, \frac{3}{5}, \frac{7}{12}, \frac{24}{41}, \dots$$

- **The 12-step scale.** The appearance of  $\frac{7}{12}$  in this list suggests that a 7-tone interval in an equally tempered scale with 12 notes per octave should be a good approximation to a perfect fifth. Computing gives that

$$2^{7/12} = 1.49830 \dots$$

This approximation is a good one: The relative error is about 1 part in 880, a small enough difference that most humans are unable to distinguish the difference between a perfect fifth and an interval of 7 steps in a 12-note equally tempered scale. In fact, the next-coarsest equally tempered scale that provides a better approximation is a 29-step scale.

Remark: The computations necessary to construct an evenly tempered scale with 12 steps in an octave were first carried out by Zhu Zaiyu, in *Complete Compendium of Music and Pitch* (1584), and Simon Stevin (1585).



Seven steps (red) vs. perfect fifth (blue) over 10 cycles of the original tone

- \* It turns out that the 12-note scale also closely represents a perfect fourth ( $2^{5/12} = 1.33484 \dots \approx \frac{4}{3}$ ) and, less precisely, a major third ( $2^{4/12} = \sqrt[3]{2} = 1.25991 \dots \approx \frac{5}{4}$ ).

# Rhythm

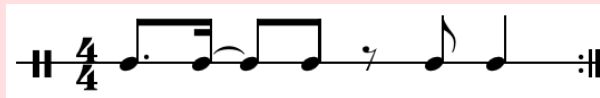
1. Rhythm adds structure to a musical composition, and arguably even more so than melodies, rhythms are mathematical structures.
2. Counting rhythms of certain types

EXAMPLE. Reinterpreting a conclusion from our example from our example concerning prosody in Sanskrit poetry from Class 5, we can conclude the following:  
The number  $a_m$  of different beat patterns of length  $m$  eighth notes consisting just of quarter notes and eighth notes is the  $(m + 1)$ st Fibonacci number,  $F_{m+1}$ .

EXERCISE. Find a recurrence formula (so, express a generic term  $a_m$  of the sequence in terms of some combination of  $a_{m-1}, a_{m-2}, \dots$ ) for the number of different beat patterns of length  $m$  eighth notes consisting just of quarter notes, eighth notes, and rests, and write down the values  $a_1, \dots, a_5$ .

## EXAMPLE (Bo Diddley Beat)

The Bo Diddley Beat is a well-known beat pattern popularized by Bo Diddley. It is in turn a variation of the *3-2 clave*, a bell pattern common in Afro-Cuban music.



Standard musical notation



Schematic using black bars to denote beats (of proportional duration) and a light bar to denote a pause

Examples of use in composition:

- Bo Diddley, *Bo Diddley* (1955)
- The Stooges, *1969* (1969)
- The Band & Ronnie Hawkins *Who Do You Love (Live)* (1976) (a cover of another Bo Diddley song)
- George Michael, *Faith* (1987)

- **Time signature.** The **time signature** is a notation to specify the number of beats in each bar and the note value equal to a beat; time signature does not determine the beat but rather encodes an aspect of it.
  - By far the most common in modern Western music is 4/4, or *common time*. (Ludwig van Beethoven, *Ninth Symphony, Movement IV (Ode an die Freude)* (1824); Martha and the Vandellas, *Dancing in the Streets* (1964))
  - The next most common is 3/4, sometimes called *perfect time*. (Piotr Ilych Tchaikovsky, *The Waltz of the Flowers* (1892); Aretha Franklin, *(You Make Me Feel Like) A Natural Woman* (1968))
  - A variant of 3/4 is the compound time signature 6/8; compositions annotated in 6/8 tend to be more upbeat than those in 3/4 and are common in genres like swing (Neutral Milk Hotel, *In the Aeroplane Over the Sea* (1998); Jay-Z, *My 1st Song* (2003))
  - Many other time signatures are used:
    - \* 5/4: Dave Brubeck, *Take Five* (1959)
    - \* 7/4: Pink Floyd, *Money* (1973)
    - \* 9/8 (but arguably 3/4 or 6/8): Jimi Hendrix, *Manic Depression* (1967)