

# MATH 110: Class 9

## August 29: Graph theory and networks; Algorithms

For Wednesday, September 1:

**Reading** *How Many Times Do I Have to Shuffle This Deck?*, David Austin, December 2010, American Mathematical Society

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### Graph theory

1. **Graphs.** Graphs are ubiquitous in applications.

- A **graph** is (1) a set  $\mathcal{V}$  of **vertices** (“nodes”) together with (2) a set  $\mathcal{E}$  of **edges**: Each edge connects 2 vertices. We sometimes denote a graph by  $(\mathcal{V}, \mathcal{E})$  and may denote it by a single letter, often  $\Gamma$ .

EXAMPLES.

- (a) Consider the graph  $\Gamma$  whose vertices are students in our class, and where 2 vertices are connected by a single edge if they have taken a course together previously and no edge otherwise.
- (b) Consider the graph  $\Gamma$  whose vertices are students in our class, and where 2 vertices are connected by a single edge if one of them would throw the ball to the other using the rule we developed on our first day.
- (c) Consider the graph  $\Gamma$  whose vertices are members of Facebook, and where 2 vertices are connected by a single edge if the 2 members are friends.

EXAMPLES (**Abstract graphs**).

- (a) (**Empty graph**) The graph with no vertices and no edges:  $\mathcal{V} = \emptyset$ ,  $\mathcal{E} = \emptyset$ .
  - (b) (**Trivial graph**) The graph  $E_1$  with 1 vertex and no edges:  $\mathcal{V} = \{*\}$ ,  $\mathcal{E} = \emptyset$ . More generally, the **edgeless graph**  $E_n$  has  $n$  vertices and no edges.
  - (c) (**Path graph**) The graph  $P_n$  with  $n$  vertices,  $1, \dots, n$ , and edges  $(1, 2), \dots, (n-1, n)$ .
  - (d) (**Cycle**) The graph  $C_n$  which is the same as the path graph but with the additional edge  $(n, 1)$ .
  - (e) (**Complete graph**) Consider the graph  $K_n$  with  $n$  vertices where there is a single edge between each pair of distinct vertices. (How many edges does  $K_n$  have?)
- Graphs are allowed to have an edge that connects to itself (a **loop**) or more than one edge between two given vertices. If we want to rule out these possibilities we ask for our graph to be **simple**. A **path** is a sequence of edges connecting successive vertices. A graph is **connected** if any two vertices are connected by a path.

EXAMPLE (**Small simple graphs**).

How many distinct simple graphs are there with 0 vertices? 1? 2? 3? Which are connected?

(As you might guess, the number of possibilities increases rapidly. An approximate formula gives that the number of distinct graphs on  $n$  vertices is approximately  $\frac{2^{\binom{n}{2}}}{n!}$ .)

ACTIVITY (**Simple graphs with 4 vertices**)

Find all simple graphs with 4 vertices. (Hint: There are 11 of them.) If you find them all, try to find all simple graphs with 5 vertices (there are 34 of them).

2. **Directed graphs (a.k.a. digraphs).** A **digraph** is a graph where each edge has a direction.

EXAMPLES

- (a) Consider the graph  $\Gamma$  whose vertices are students in our class, and where there is a directed edge from  $A$  to  $B$  if the rule we developed on our first day would have  $A$  throw the ball to  $B$ .
- (b) Consider the graph  $\Gamma$  whose vertices are Web pages, and where there is a directed edge from  $A$  to  $B$  if  $A$  contains a link to  $B$ . (Google’s search engine relies on analysis of this graph.)
- (c) Consider the graph  $\Gamma$  whose vertices are members of Instagram, and where there is a directed edge from  $A$  to  $B$  if  $A$  follows  $B$ .

3. **Weighted graphs.** A **weighted graph** is a graph where each edge has (usually numerical) weight assigned to it, equivalently, a function  $\mathcal{E} \rightarrow \mathbb{R}$ .

#### EXAMPLES

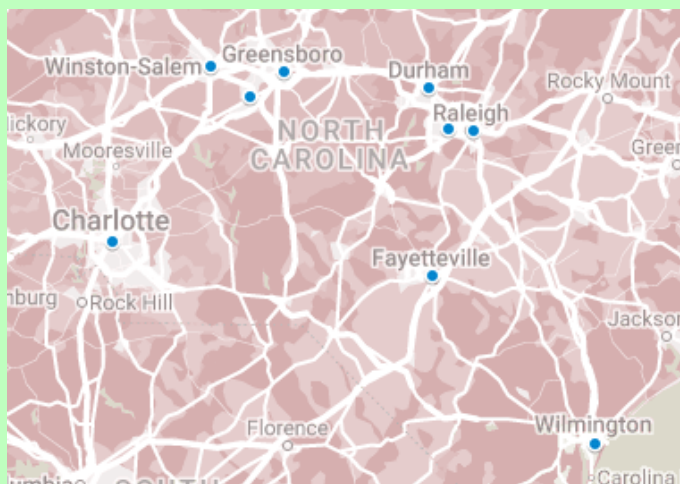
(a) Consider the weighted graph  $\Gamma$  whose underlying graph is the complete graph on the set  $\mathcal{V}$  of incorporated places in North Carolina and the weight of the edge  $(A, B)$  is the distance from  $A$  to  $B$  (say, in km).

- **The Traveling Salesman Problem.** Given a weighted graph  $\Gamma$  whose underlying graph is complete, find the cycle that includes every vertex (such a cycle is called a **Hamiltonian cycle**) with the smallest sum of weights. (In our previous example, find the route that starts and ends in the same city of the shortest length.) How many possible routes are there, i.e., how many Hamiltonian cycles are there in a complex graph on  $n$  vertices?

**ACTIVITY** (Most populous places in North Carolina) North Carolina's 9 largest cities are, in decreasing order, Charlotte, Raleigh, Greensboro, Durham, Winston-Salem, Fayetteville, Cary, Wilmington, and High Point.

The distances between these cities are given by the following table:

	<i>Charlotte</i>	<i>Raleigh</i>	<i>Greensboro</i>	<i>Durham</i>	<i>Winston-Salem</i>	<i>Fayetteville</i>	<i>Cary</i>	<i>Wilmington</i>	<i>High Point</i>
Charlotte	—	274	148	230	127	212	261	328	124
Raleigh		—	130	47	172	101	19	211	154
Greensboro			—	87	47	151	114	336	27
Durham				—	129	143	31	253	111
Winston-Salem					—	191	158	380	32
Fayetteville						—	116	143	167
Cary							—	225	140
Wilmington								—	341



Find the shortest round-trip distance that you can that visits all 9 cities.