

MATH 110: Class 08

August 26: Euclidean geometry in 3 and more dimensions

For Monday, August 29:

Reading Having Fun with the 4-Color Theorem, Evelyn Lamb, *Scientific American*, 2013 March 1.

For Tuesday, August 31:

Homework Problem Set 2 due

Euclidean geometry in 3 dimensions

1. **Additional constructions.** In 3 dimensions, we have more room, and therefore more constructions than we had in 2-dimensional Euclidean geometry.

- A **plane** is a flat, 2-dimensional surface that extends infinitely far in all directions.
 - Just as a line can be determined by 2 distinct points, a plane is determined by three noncollinear points.
- A **sphere** is the analogue of a circle: It consists of all points a fixed distance from a given point.
- A **polyhedron** (from the Greek *πολυς* *many* + *εδρα* *seat*) is a solid whose faces are all polygons.

2. **Platonic solids.** What are the analogues in 3 dimensions of the regular polygons?

- **Schläfli symbols of regular polygons.** The **Schläfli symbol** of a regular n -gon is $\{n\}$.

EXAMPLE (Cube). A cube is a solid shape whose faces are 6 squares, i.e., regular 4-gons; 3 squares meet at each vertex. We thus denote this polyhedron by its *Schläfli symbol*, $\{4, 3\}$, 4 for the numbers of sides of each face, and 3 for the number of those faces meeting at any vertex.

What are the possible Schläfli symbols $\{p, q\}$ for **convex, regular polyhedra**, that is, convex polyhedra whose faces are copies of the same regular p -gon with q faces meeting at a corner? The sum of the angles of all of the faces meeting at a point must be $< 360^\circ$. The angle at a vertex of a regular p -gon is $180^\circ \times (1 - \frac{2}{p})$. So, we must have

$$180^\circ \times \left(1 - \frac{2}{p}\right) \times q < 360^\circ,$$

or, equivalently,

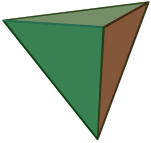
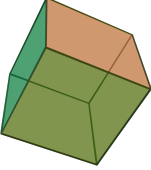
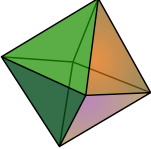
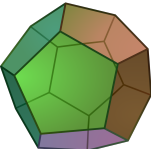
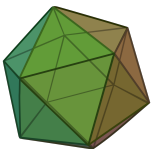
$$q < \frac{2p}{p-2}$$

Since we're interested in producing solid figures, we must have $p \geq 3$. Likewise, we must have $q \geq 3$, which implies $p < 6$.

This leaves five possible convex, regular polyhedra:

$\{3, 3\}$	3 equilateral triangles meet at each corner
$\{4, 3\}$	3 squares meet at each corner (we recognize this shape as the cube)
$\{3, 4\}$	4 equilateral triangles meet at each corner
$\{5, 3\}$	3 regular pentagons meet at each corner
$\{3, 5\}$	5 equilateral triangles meet at each corner

- **The five Platonic solids.** It turns out that all five possible Schläfli symbols correspond to actual shapes; these are the **Platonic solids**.

	name	symbol	faces	edges	vertices
	tetrahedron	$\{3, 3\}$			
	cube	$\{4, 3\}$			
	octahedron	$\{3, 4\}$			
	dodecahedron	$\{5, 3\}$			
	icosahedron	$\{3, 5\}$			

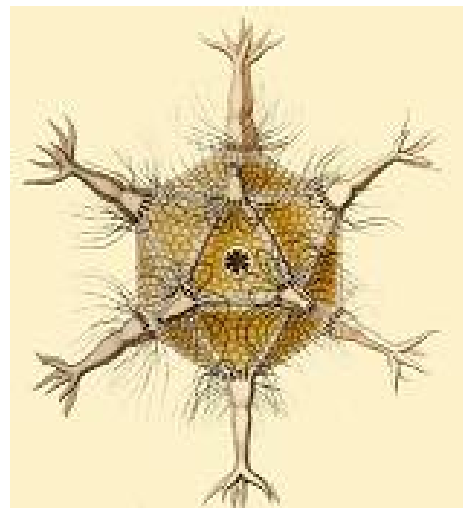
EXERCISE.

- Complete the table above.
- Where do any or all of these solids arise in applications?

- **Visualizing polyhedra.** Projections and wireframes.



Roman dodecahedron, 2nd–4th Century, Tongeren



The radiolarian species *Circogonia icosahedra*

- **Dual regular polyhedra.** The **dual** of a regular polyhedron with symbol $\{p, q\}$ is the regular polyhedron with symbol $\{q, p\}$.

EXERCISE (Duals of the Platonic solids). What is the dual of each of the five Platonic solids? What is the relationship between the counts of faces, edges, and vertices of a Platonic solid and those of its dual?

- **Euler characteristic.** The **Euler characteristic** $\chi(P)$ of a polyhedron P with F faces, E edges, and V vertices is $\chi(P) = F - E + V$.

EXERCISE. What is the Euler characteristic of each Platonic solid?

- How would a 2-dimensional being confined to a plane see a solid cutting through that plane?

- **EXERCISE.** Sketch cross-sections of spheres and Platonic solids.

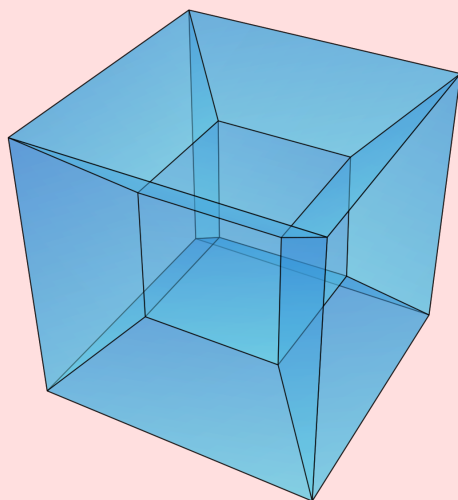
Euclidean geometry in 4 and more dimensions

- In 4-dimensional Euclidean space we have 4 degrees of freedom of movement. To complement the existing cardinal directions (left, right, up, down, forward, back) Charles Hinton proposed the words *ana* and *kata*.
- 3-sphere.** The 3-sphere of radius r centered at O is the set of all points in 4-dimensional Euclidean space. The 3-dimensional cross-sections of the 3-sphere are spheres.
- Polychora.** A **polychoron** (Greek $\chi\omega\rho\omicron\varsigma$ *room*) (**4-polytope**) is a 4-dimensional analogue of a polyhedron.
 - Its *facets* (“sides”) or *cells* are polyhedra.
 - Its *ridges* or *faces* are polygons.
 - Its *peaks* or *edges* are line segments.

EXAMPLE (Tesseract).

A **tesseract** (**4-cube**, [**4-dimensional**] **hypercube**, **8-cell**], [**regular**] **octachoron**) is a regular polychoron whose facets are 8 cubes, with 3 cubes meeting at each edge. So, we denote it by the Schläfli symbol $\{4, 3, 3\}$.


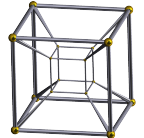
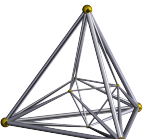
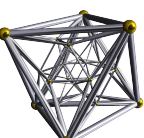
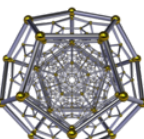
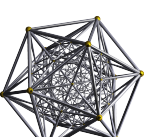
We can draw its edges and project it to a wireframe. Draw a point, line segment, square, cube, and tesseract, each by “extending” the previous object.



EXAMPLE (Regular 4-simplex).

A **regular 4-simplex** (**pentachoron**, [**regular**] **5-cell**) is a regular polychoron whose facets are 5 tetrahedra, with 3 tetrahedra meeting at each edge. So, its Schläfli symbol is $\{3, 3, 3\}$.

- The regular, convex polychora.

	name	symbol	analogous polyhedron
	4-simplex	$\{3, 3, 3\}$	tetrahedron
	tesseract	$\{4, 3, 3\}$	cube
	16-cell	$\{3, 3, 4\}$	octahedron
	24-cell	$\{3, 4, 3\}$	—
	120-cell	$\{5, 3, 3\}$	dodecahedron
	600-cell	$\{3, 3, 5\}$	icosahedron

4. **Higher dimensions: Polytopes.** The notion *polytope* (Greek *τοπος place*) generalizes polyhedron, etc., to arbitrary dimension. An n -**polytope** is a polytope in n dimensions, so polygon, polyhedron, polychoron, are respectively synonyms for 2-polytope, 3-polytope, 4-polytope.

- **Regular polytopes.** In each dimension $n > 4$, there are exactly three regular polytopes.

name	symbol	analogous polyhedron
n -simplex	$\{3, \dots, 3\}$	tetrahedron
n -tesseract	$\{4, 3, \dots, 3\}$	cube
n -orthoplex	$\{3, \dots, 3, 4\}$	octahedron