

# MATH 110: Class 05

## August 23: Functions, Sequences and pattern-finding

For Wednesday, August 24:

**Reading** How to Quantify (and Fight) Gerrymandering, Erica Klarreich, *Quanta Magazine*, 2017 April 4.

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## Functions

1. **Definitions and notation** : The notation for the element of  $Y$  assigned to  $x \in X$  is  $f(x)$ .
  - A **function** (or **map**)  $f$  from a set  $X$  to a set  $Y$  is a rule that assigns to each element of  $X$  exactly one element of  $Y$ .
  - We denote a function  $f$  from  $X$  to  $Y$  by  $f : X \rightarrow Y$ .
  - We denote the element of  $Y$  to which  $f$  sends a given  $x \in X$  by  $f(x)$ .
  - For a function  $f : X \rightarrow Y$ , we call  $X$  the **domain** of  $f$ ,  $Y$  the **codomain** of  $f$ , and the set  $f(X) = \{f(x) \mid x \in X\}$  (that is, the set of all values in  $Y$  that  $f$  takes) the **image** of  $f$ .

EXAMPLE. The following are functions.

- (a) **(The increment function)**  $X = Y = \mathbb{N}$ ,  $f(x) = x + 1$
- (b) **(The squaring function)**  $X = Y = \mathbb{Z}$ ,  $f(x) = x^2$  (we can also view this as a map  $\mathbb{Z} \rightarrow \mathbb{N}$ ;
- (c) **(The identity function)** Any set  $X = Y$ ,  $\text{id}_X(x) = x$
- (d) **(Complex conjugation)**  $X = Y = \mathbb{C}$ ,  $f(z) = \bar{z}$
- (e) **(Constant functions)**  $X$  and  $Y$  are any set, and  $y_0 \in Y$ :  $c_{y_0}(x) = y_0$
- (f)  $X = \{1, 2, 3\}$ ,  $Y = \{A, B, C\}$ ,  $f(1) = f(2) = A$ ,  $f(3) = B$
- (g)  $X = \{\text{students in our class}\}$ ,  $Y = \{\text{January}, \dots, \text{December}\}$ ,  $f(x) = x$ 's birth month
- (h)  $X = \{\text{all possible orderings of a standard set of playing cards}\}$ ,  $Y = \{\text{standard playing cards}\}$ ,  $f(x) =$  the top card of the deck in ordering  $x$
- (i)  $X = \{\text{cars currently registered in North Carolina}\}$ ,  
 $Y = \{\text{the set of all possible strings of characters on N.C. license plates}\}$ ,  
 $f(x) =$  the license plate of car  $x$ .

EXERCISE. What other functions can you think of?

2. **Properties of particular functions.**

- A function  $f : X \rightarrow Y$  is **one-to-one** (or **injective**) if there are no distinct  $x, x' \in X$  such that  $f(x) = f(x')$ .
- A function  $f : X \rightarrow Y$  is **onto** (or **surjective**) if for every  $y \in Y$  there is at least one  $x \in X$  such that  $f(x) = y$ . Put another way, the image  $f(X)$  is the entire codomain  $Y$ .
- A function  $f : X \rightarrow Y$  is **bijective** if it is both **one-to-one** and **onto**.

EXERCISE. Which of the functions we've looked at are one-to-one? Onto? Bijective?

3. **Function composition.** If  $f$  and  $g$  both have codomain  $\mathbb{R}$ , we can form a new function, the **composition** of  $f$  and  $g$ , which we denote by  $f \circ g$  and define by

$$(f \circ g)(x) = f(g(x)).$$

4. **Associativity of function composition.** Function composition is associative. More precisely, for any maps  $f : W \rightarrow X$ ,  $g : X \rightarrow Y$ , and  $h : Y \rightarrow Z$ , we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

This identity shows that we can write the composition of three or more functions without parentheses, e.g., as  $f \circ g \circ h$ , without creating ambiguity.

5. **Inverse functions.** If a function  $f : X \rightarrow Y$  is bijective, there is a unique map  $f^{-1} : Y \rightarrow X$ , called the **inverse** of  $f$ , that undoes what  $f$  does and vice versa, i.e., satisfying  $f^{-1}(f(x)) = x$  and  $f(f^{-1}(y)) = y$ .
6. **Sequences revisited.** We can think of a sequence  $(a_i)$  of objects in a set  $X$  as the function  $\mathbb{N} \rightarrow X$  defined by  $i \mapsto a_i$ .
7. **The Pigeonhole Principle revisited.** In the language of functions, the Pigeonhole Principle says that if the domain and codomain of a function are both finite and there are more elements in the domain than in the codomain, then  $f$  cannot be one-to-one.
8. **Aside: Cartesian products.** For sets  $X$  and  $Y$  the **Cartesian product** of  $X$  and  $Y$  is the set

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

9. **Using functions and operations to build new functions** If both  $f$  and  $g$  have domain  $X$  and codomain  $\mathbb{R}$ , we can form new functions  $X \rightarrow \mathbb{R}$ :

- $(f + g)(x) = f(x) + g(x)$
- $(f - g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$

More generally, if  $f$  and  $g$  are both functions  $X \rightarrow Y$  and  $*$  is a binary operator on  $Y$ , we can define another function

$$f * g : X \rightarrow Y, \quad (f * g)(x) = f(x) * g(x).$$

10. **The graph of a function.** The **graph** of a function  $f : X \rightarrow Y$  is the set

$$\Gamma(f) = \{(x, y) \mid y = f(x)\},$$

which by construction is a subset of the Cartesian product  $X \times Y$ . If  $X$  and  $Y$  are both subsets of  $\mathbb{R}$ , then we can think of the graph as a subset of the Cartesian plane.

11. **Some important real-valued functions defined on (subsets of)  $\mathbb{R}$**

- **Polynomial functions.**

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

- If  $a_n \neq 0$ , we say that  $f$  has **degree**  $n$ .
- **(constant functions)**  $p(x) = c$
- **(linear functions)**  $p(x) = ax + b$
- **(quadratic functions)**  $p(x) = ax^2 + bx + c$

- **Rational functions.** Ratios of polynomial functions  $p, q$ .

$$f(x) = \frac{p(x)}{q(x)}$$

- **$n$ th root functions.**

$$f(x) = \sqrt[n]{x}$$

- **Exponential functions.**

$$f(x) = Ab^x$$

- **Logarithmic functions.**

$$\log_b$$

- **Trigonometric functions.**

$$\sin, \quad \cos, \quad \tan, \quad \sec, \quad \csc, \quad \cot$$

- **Inverse trigonometric functions.**

$$\arcsin, \quad \arccos, \quad \arctan, \quad \operatorname{arcsec}, \quad \operatorname{arccsc}, \quad \operatorname{arccot}$$

## Binary operations

1. **What is a binary operation?** A binary operation  $*$  on a set  $S$  is a rule that takes two elements of  $S$  and returns one element of  $S$ ; put another way, a binary operation is a function  $X \times X \rightarrow X$ . We might denote an operation with the notation

$$*: S \times S \rightarrow S.$$

2. **Why do we care about binary operations?**

EXAMPLE.

- (a) **(integer addition)**  $S = \mathbb{Z}$  (here  $\mathbb{Z}$  denotes the integers, i.e., natural numbers along with their negatives) with operation  $+$
- (b) **(integer subtraction)**  $S = \mathbb{Z}$  with operation  $-$
- (c) **(integer multiplication)**  $S = \mathbb{Z}$  with operation  $\times$
- (d) **(integer distance)**  $S = \mathbb{Z}$  with the operation  $*$  that takes two integers and gives the distance between them, that is,  $x * y = |x - y|$
- (e) **(integer maximum)**  $S = \mathbb{Z}$  with operation  $*$  that takes two integers and returns whichever of them is largest.
- (f) **(concatenation)**  $S$  is the set of all strings made up of symbols from some character set (say, the set of all text messages using standard Latin characters and punctuation) with the **concatenation** operation  $\circ$ , where  $x \circ y$  is the single message whose contents are all of  $x$  followed immediately by all of  $y$ .
- (g) **(Rock, Paper, Scissors)**  $A = \{R, P, S\}$  with the operation  $\star$  defined by taking  $x \star x = x$  for any  $x \in S$  and otherwise declaring  $x \star y$  to be the winner of the match  $x$  vs.  $y$  according to the rules of *R(ock)*, *P(aper)*, *S(cissors)*
- (h) **(operations on power sets—a little advanced!)** Given any set  $X$ ,  $\mathcal{P}(X)$  (the **power set** of  $X$ ) denotes the set of all subsets of  $X$ . Then,  $(\mathcal{P}(X), \cup)$ ,  $(\mathcal{P}(X), \cap)$ , and  $(\mathcal{P}(X), \setminus)$ . Is  $(\mathcal{P}(X), \cdot^c)$  a binary operation on  $\mathcal{P}(X)$ ? Why or why not?

EXERCISE. What are some other binary operations? For each be sure to specify both the underlying set  $S$  and the operation  $*$ .

3. **Operation tables.** Recall that we often make tables of the above examples, especially multiplication. We can likewise make a table for any operation. If  $S$  has finitely many elements, we can use a table to specify an operation.

EXAMPLE.

(a) **(Rock, Paper, Scissors).**

*	R	P	S
R	R	P	R
P	P	P	S
S	R	S	S

(b) **(Addition modulo 2)** (This is a special case of *modular addition*, which we'll see later in the term.)

+ <sub>2</sub>	0	1
0	0	1
1	1	0

(c) **(Multiplication modulo 2)** (Analogously this is a special case of *modular multiplication*.)

× <sub>2</sub>	0	1
0	0	0
1	0	1

(d) What do addition and multiplication modulo 2 have to do with binary representations of numbers?

4. **(Two-sided) identities.** Given an operation  $*$  on a set  $S$ , the element  $e \in S$  is a **(two-sided) identity** (or **identity element** or **neutral element**) if, for all  $s \in S$ ,  $e * s = s$  and  $s * e = s$ , that is if using our operation to combine any  $s \in S$  with  $e$  (in either order) always gives back  $s$ .

EXERCISE.

- Which of the operations  $(S, *)$  that we've looked at have an identity; in those cases what are the identity elements? Which of the operations that we've looked at don't?
- What does the row or column of the identity element in a multiplication table look like?

## 5. Properties of operations

- Recall that the order in which we add or multiply two integers (or real numbers) doesn't affect the value of the sum, e.g.,  $1 + 2 = 2 + 1$ . We call this property **commutativity** of the operation; more precisely, we say that the operation  $(S, *)$  is **commutative** if  $x * y = y * x$  for all  $x$  and all  $y$  in  $S$ .

EXERCISE.

- Which of the operations we've looked at are commutative, and which are not?
- What does the multiplication table of a commutative operation look like?

- Likewise, if we add three numbers by adding two of them and then adding the third, it doesn't matter which two we add first, that is,  $(x + y) + z = x + (y + z)$ —so that writing  $x + y + z$  is unambiguous. We call this property **associativity** of the operation; more precisely, we say that the operation  $(S, *)$  is **associative** if  $(x * y) * z = x * (y * z)$ ; if  $(S, *)$  is associative, we can unambiguously write  $x * y * z$ .

EXERCISE. Which of the operations we've looked at are associative, and which are not?

## (Propositional) Logic

- Propositional logic is a branch of logic that deals with propositions (i.e., logical statements), whether they are true (1, T,  $\top$ ) or false (0, F,  $\perp$ ), and the relationships among them.
- Logical connectives**
  - Material implication** ( $\Rightarrow$ , IMPLY)

EXAMPLE

Consider the proposition “If it is raining outside, then it is cloudy.”

- (a) Provided that the statement is true, can we conclude anything if we know that it is not cloudy? If so, what? If not, why not?
- (b) Provided that the statement is true, can we conclude anything if we know that it is not raining outside? If so, what? If not, why not?

- We’d like to be able to express logical relationships with symbols instead of just words—not least so that the logical relationships we write down are easier to read than those we read in Aristotle’s *Prior Analytics*. In general we denote simple propositions by single uppercase letters.

EXAMPLE

Denote the following propositions by the given symbols:

$P$	It is raining outside.
$Q$	It is cloudy.

Then, we can write our statement as “If  $P$  is true, then  $Q$  is true,” or more concisely, “ $P$  implies  $Q$ .”

- In the proposition “ $P$  implies  $Q$ ”, “implies” is a *logical connective*, i.e., a statement about the logical relationship between  $P$  and  $Q$ . The implication (i.e., if-then) relationship is so common in logic that we give it its own symbol,  $\Rightarrow$ , and we write our proposition as “ $P \Rightarrow Q$ ”. By definition, *logical connectives* let us take one or more propositions and build a new proposition from them. This should remind you of binary operations—which is not a coincidence!
- Remark: We call propositions that have no logical connectives **atomic propositions** (the word “atomic” comes from the Greek word for “indivisible”). A proposition built out of atomic propositions and one or more logical connectives a **compound proposition**.

• **Negation** ( $\neg$ , NOT,  $\sim$ )

- Given any proposition  $P$ , we can form another proposition, namely “the proposition  $P$  is not true”. We denote this new proposition symbolically by “ $\neg P$ ”. If  $P$  is true, then  $\neg P$  is false, and vice versa. If we let 0 denote the proposition that is always true and 1 the proposition that is always false, then

$$\neg 0 = 1 \quad \text{and} \quad \neg 1 = 0.$$

Here, we say that two propositions  $A$  and  $B$  are equal, and write  $A = B$ , if  $A$  is true whenever  $B$  is, and vice versa. The above equations say in particular that we can think of  $\neg$  as a *unary operator* on the set  $\{0, 1\}$ , that is, an operator that takes *one* element of  $\{0, 1\}$  and gives back an element of  $\{0, 1\}$ . Notice that for any proposition  $P$ ,

$$\neg \neg P = P.$$

This statement can be interpreted as saying that “saying that it’s not true that  $P$  isn’t true is the same as saying that  $P$  is true”.

EXERCISE

If  $P$  and  $Q$  are the propositions in our running example, how would we describe the propositions  $\neg P$  and  $\neg Q$  in words?

- In our running example we concluded that if it’s true that “If it’s raining, then it is cloudy”, then it’s also true that “If it’s not cloudy, then it’s not raining.” We can write this observation in terms of the operators  $\Rightarrow$  and  $\neg$ . In terms of propositions  $P$  and  $Q$ , we say that if the proposition  $P \Rightarrow Q$  is true, then  $(\neg Q) \Rightarrow (\neg P)$  is also true. Better yet:

$$(P \Rightarrow Q) \Rightarrow ((\neg Q) \Rightarrow (\neg P)).$$

The proposition  $(\neg Q) \Rightarrow (\neg P)$  is called the **contrapositive** of the proposition  $P \Rightarrow Q$ : The previous sentence says that if a proposition is true, so is its contrapositive.

- Notice that if the contrapositive  $(\neg Q) \Rightarrow (\neg P)$  is true, then so is its own contrapositive (that is, the contrapositive of the contrapositive of  $P \Rightarrow Q$ ), namely,  $(\neg(\neg P)) \Rightarrow (\neg(\neg Q))$ , but by our observation

in the previous bullet point, this is just  $P \Rightarrow Q$ , that is, the contrapositive of the contrapositive of a proposition is just the same proposition again. In particular, this tells us that a proposition is true if and only if its contrapositive is true.

- **Conjunction** ( $\wedge$ , AND, &, &&,  $\cdot$ )

- We define the **conjunction**  $\wedge$  by declaring that  $P \wedge Q$  is true if and only if  $P$  is true and  $Q$  is true. This makes  $\wedge$  a binary operator on the set  $\{0, 1\}$ . Its multiplication table is thus

$\wedge$	0	1
0	0	0
1	0	1

- We can just as well record this operator using a *truth table*:

$P$	$Q$	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

**EXERCISE**

If  $P$  and  $Q$  are the propositions in our running example, how would we describe the proposition  $P \wedge Q$  in words?

- Remark: Notice that  $\wedge$  is the same as the binary operation “multiplication modulo 2” that we yesterday denoted  $\times_2$ , so in fact  $\wedge$  and  $\times_2$  are the same operation!

- **Inclusive disjunction** ( $\vee$ , OR, |, ||, +)

- We define the **inclusive disjunction**  $\vee$  by declaring that  $P \vee Q$  is true if and only if  $P$  is true or  $Q$  is true (or both are true).

**EXERCISE.**

- Write the operation table and the truth table for the operation  $\vee$
- If  $P$  and  $Q$  are the propositions in our running example, how would we describe the proposition  $P \vee Q$  in words?

- In fact, we can define  $\vee$  in terms of  $\wedge$  by observing that  $P \vee Q = \neg((\neg P) \wedge (\neg Q))$ ; this identity is one of *de Morgan's Laws* (the other is  $P \wedge Q = \neg((\neg P) \vee (\neg Q))$ ).

- **Implication revisited**

- If we think of  $\Rightarrow$  as a binary operation on  $\{0, 1\}$ , how is it defined?

$\Rightarrow$	0	1
0	1	1
1	0	1

- Think of the proposition  $P \Rightarrow Q$  as a promise that if  $P$  happens then so does  $Q$ : The promise is only broken if  $P$  happens but  $Q$  does not.

**EXERCISE.** Write the truth table for the operation  $\Rightarrow$ .

- **Other logical connectives**

- We can use  $\Rightarrow$ ,  $\neg$ ,  $\wedge$ , and  $\vee$  to produce other logical connectives. For example:
  - \* the biconditional operator  $\Leftrightarrow$  (**XNOR**,  $=$ ,  $\equiv$ ):  $P \Leftrightarrow Q = ((\neg P) \wedge (\neg Q)) \vee (P \wedge Q)$ , which is true if and only if  $P$  and  $Q$  are both true or both false.
  - \* exclusive disjunction,  $\underline{\vee}$  (**XOR**):  $P \underline{\vee} Q = (P \wedge (\neg Q)) \vee ((\neg P) \wedge Q)$  is true if and only if one of  $P$  and  $Q$  is true and the other is false (so we also have  $P \underline{\vee} Q = \neg(P \Leftrightarrow Q)$ ). Notice that  $\underline{\vee}$  and

the binary operation “addition modulo 2” that we yesterday denoted  $+_2$  that we saw yesterday are the same!

EXERCISE.

- (a) If we think of  $\Rightarrow$ ,  $\wedge$ ,  $\vee$ ,  $\Leftrightarrow$ ,  $\perp$  as binary operators, which are commutative and which are not? Which are associative?
- (b) How many different logical connectives (of two propositions) are there? Put another way, how many possible binary operations are there on the set  $\{0, 1\}$ , or yet another way, how many different ways are there to fill in a blank operation table on  $\{0, 1\}$ ?

- **The modus ponens**

- If  $P$  is true and  $P$  implies  $Q$ , then  $Q$  is also true:

$$(P \wedge (P \Rightarrow Q)) \Rightarrow Q.$$

- Closely related is the notion of Socratic syllogism, the standard example of which is:
  - \* All men are mortal.
  - \* Socrates is a man.
  - \* Therefore, Socrates is mortal.