

MATH 110: Class 02

August 18

For Monday, August 22:

Project First project check-in: Come with proposal for project topic

Algebra

1. What is algebra?

- Solving equations for unknowns, that is finding the value(s) of one or more variables that make an equation true. If evaluating an expression is “forward thinking”, then solving algebraically is “backward thinking”.
- Why do we care about algebra?

2. The beginnings of algebra: al-Kwārizmī, *Al-Jabr*, and Omar Khayyam.

- Algebra was first conceptualized as its own topic of study by al-Kwārizmī (c. 780–c. 850)¹, a Persian polymath who wrote extensively about mathematics, astronomy, and cartography, and who served as head library and astronomer at the House of Wisdom in Baghdad, in his treatise, *The Compendious Book on Calculation by Completion and Balancing* (c. 813–c.833). The Arabic title is *al-Kitāb al-Mukhtaṣar fī Ḥisāb al-Jabr wal-Muqābalah*, sometimes shortened to *Al-Jabr*, from when we derive the word “algebra”, which by itself means “the mending (of broken pieces)”. The text was translated into Latin in the 12th C., and it remained the principal mathematics text in Europe until the 16th C. It’s difficult to overstate the importance of this book to later developments in mathematics, which conceptualized rational numbers, irrational numbers (both of which we’ll talk about tomorrow) and geometric measurements as algebraic objects in their own right.
- Also a renowned astronomer and poet, Omar Khayyam’s greatest mathematical achievements are his study of Euclid’s Parallel Postulate, setting the stage for the study of Noneuclidean geometries much later, and his work on solving cubic (third-degree polynomial) equations. He wrote:

Whoever thinks algebra is a trick in obtaining unknowns has thought it in vain. No attention should be paid to the fact that algebra and geometry are different in appearance. Algebras are geometric facts which are proved by Propositions 5 and 6 of Book II of [Euclid’s] *Elements*.

3. Doing algebra

- There are many techniques for solving different kinds of equations, and the most basic and most common one—emphasized in the *Al-Jabr* takes advantage of the fact that performing the same operation on two equal quantities yields again equal quantities.

EXAMPLE. Solve the following algebraic equations:

- (a) $x + 1 = 3$
- (b) $2x - 1 = 5$
- (c) $x^2 = 4$
- (d) $x^2 + 3x + 2 = 0$

EXERCISE. Solve the following algebraic equations:

- (a) $x - 2 = 4$
- (b) $3x + 1 = 7$
- (c) $x^2 = 9$
- (d) $x^2 - 6x + 8 = 0$

¹al-Kwārizmī’s name is also the origin of the English word *algorithm*.

- Much of the power of algebra comes from its ability to manipulate unknown or abstract values—essentially as easily as it does concrete numbers. As a consequence, we can use algebra to solve all problems of a given form at the same time.

EXAMPLE (Solving a quadratic equation in vertex form) Suppose $a \neq 0$. Solve the equation

$$a(x - h)^2 + k = 0$$

for x .

EXERCISE (Converting the equation of a line in standard form to slope-intercept form) Suppose $b \neq 0$. Solve the equation

$$ax + by = c$$

for y .

- **Deriving the Quadratic Formula**

Sets

1. What is a set?

- Essentially a set is a collection of mathematical objects, often bound by some commonality. We often write sets as lists inside braces. They can be finite (i.e., contain finitely many objects) or infinite (respectively, infinitely many). An object x in a set S is called x an **element** of S , which we denote by $x \in S$. If an object y is not in S , we write $x \notin S$. If every element of a set S is also in the set T , we write $A \subseteq T$ and say, “ S is a **subset** of T ”.
- **Why do we care about sets?**

EXAMPLE.

(a) The set $\{1, 2, 3\}$ is the set exactly containing the numbers 1, 2, and 3.

- This set is the same as $\{1, 3, 2\}$, $\{2, 1, 3\}$, etc.: The order in which we write down the elements of a set is irrelevant.
- This set is the same set as $\{1, 1, 2, 3\}$ —an object is either in a given set or it isn’t, there’s no such thing as “being in a set twice”.

(b) **(The empty set)** The (unique) set that contains no elements is the **empty set**, which we denote $\emptyset = \{\}$.

(c) **(The natural numbers)** When we talk about a set that contains infinitely many numbers, we might indicate enough terms to indicate a pattern and use an ellipsis (“...”) to indicate that it continues forever. For example, we might write the set of nonnegative integers (a.k.a. *natural numbers*) as

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}.$$

What are the disadvantages of this notation?

2. **Set-builder notation.** We can emphasize the defining characteristic of elements, often save writing, and eliminate ambiguity using **set-builder notation**.

EXAMPLE. We might describe the natural numbers as

$$\mathbb{N} = \{x : x \text{ is a positive integer}\}.$$

More generally, if $P(x)$ is a true-false statement that depends on x , the notation

$$\{x : P(x)\}$$

means “the set of all x such that the statement $P(x)$ is true.”

EXAMPLE. Another variation on this idea uses one set to describe another. What elements are in the set

$$\{x^2 : x \in \mathbb{N}\}?$$

EXERCISE. Use set-builder notation to describe the following infinite sets:

- (a) **(even, positive integers)** $\{2, 4, 6, 8, 10, \dots\}$
- (b) **(powers of 2 with natural exponent)** $\{1, 2, 4, 8, 16, \dots\}$
- (c) **(Egyptian fractions)** $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$

3. Cardinality

- We say that two sets have the same size, or more precisely, that they have the same **cardinality** if there is a one-to-one correspondence between the sets, that is, if we can match up items in the first set one-to-one with items in the second set without missing any items.

EXAMPLE. The sets $\{1, 2, 3\}$ and $\{\Delta, \square, \circ\}$ have the same cardinality (namely, 3).

- Does the set \mathbb{N} of natural numbers have the same cardinality as the set $2\mathbb{N}$ of even natural numbers? (Remark: We denote the cardinality of \mathbb{N} by \aleph_0 , which we read as “aleph-nought”.)
- Do all infinite sets have the same cardinality? (We’ll see tomorrow.)

4. Some operations on sets

- **(Union)** The **union** $A \cup B$ of the sets of A and B is the set of all elements in at least one of A and B ,

$$A \cup B = \{x : x \in A \text{ and/or } x \in B\}.$$

- **(Intersection)** The **intersection** $A \cap B$ of the sets of A and B is the set of all elements in *both* A and B ,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

- **(Set difference)** The **set difference** $A \setminus B$ of A and B is

$$A \setminus B = \{x : x \in S \text{ and } x \notin B\}.$$

- **(Complement)** If we have some set S (called a **universe**), then the complement of the subset $A \subset S$ in S is

$$A^c = S \setminus A$$



Loew's \aleph_0 -plex, a throwaway joke in *Futurama*, episodes 2ACV08 and 3ACV15.

Sequences and pattern-finding

1. **Sequences.** A **sequence** is an ordered list of objects and may be finite or infinite (in which case it has a first number).

2. **EXAMPLE** Identify the pattern in each of the following sequences:

- (a) $0, 1, 2, 3, 4, \dots$
- (b) $1, 2, 4, 8, 16, \dots$
- (c) $0, 0, 0, 0, 0, \dots$
- (d) $1, -3, 9, -27, 81, \dots$
- (e) $1, 3, 6, 10, 15, \dots$
- (f) $0, 3, 9, 21, 45, \dots$

3. **Caution:** We should be wary of guesses when we've only seen a few numbers of the sequence. Consider sequence (b) above and the **Lazy Caterer's Sequence**, $1, 2, 4, 7, 11, \dots$

4. **Specifying functions.** We have at least two ways to express sequences:

- Explicitly as a function $a_n = f(n)$.
- Inductively as a function $a_n = f(a_{n-1})$ or $a_n = f(a_{n-1}, a_{n-2})$, etc.

5. We can think of a sequence of objects in a set X as a function $f : \mathbb{N} \rightarrow X$.

6. **Fibonacci sequence.**

- In the 2nd or 3rd Century B.C.E. the Sanskrit poet-mathematician Pingala was interested in (and found a rule for) enumerating all the different patterns of a given length built from short syllables (length 1 unit) and long syllables (length 2 units).

length	count	patterns
1	1	■
2	2	■ ■, ■■
3	3	■ ■ ■, ■ ■■, ■■■
4	5	■ ■ ■ ■, ■ ■ ■■, ■ ■■■ ■, ■ ■■■ ■, ■■■ ■■
5	8	■ ■ ■ ■ ■, ■ ■ ■ ■■, ■ ■ ■■■ ■, ■ ■ ■■■ ■, ■■■ ■ ■■, ■■■ ■■■, ■■■ ■■■, ■■■ ■■■■
⋮	⋮	⋮

- What number comes next?
- How do we find a general pattern?
- **Definition.** Define

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \text{ for } n > 1.$$

The sequence starts thusly:

$$0, 1, 1, 2, 3, 5, 8, \dots$$

- Named for Leonardo of Pisa, who described the sequence in 1202.
- Fibonacci numbers occur frequently in applications (computer science) and nature, e.g., the arrangement of leaves on a stem or the flowering of an artichoke.
- Tessellating squares diagram; applications to the Golden Ratio ϕ /Classical aesthetics. What happens to the ratios F_{n+1}/F_n as n gets large?
- Counting the number of genetic contributors to a typical biological male's X chromosome in each previous generation.
- Tiling a $2 \times n$ board with dominos.

7. **ACTIVITY.** Create patterns of numbers of your own, write out the first several numbers in the sequence, give those first numbers to neighbors/your class, and have them try to find your rule.
8. **EXAMPLE** Some sequences whose rules are challenging to find:
- (a) **Lazy Caterer's sequence** 1, 2, 4, 7, 11, 16, 22, 29, 37, 46
 - (b) **See-and-say sequence** 1, 11, 21, 1211, 111221, 312211, ...
 - (c) **Values of the Euler totient function** 1, 1, 2, 2, 4, 2, 6, 4, 6, 4, ... (Hint: Think about factors of natural numbers.)
 - (d) **Catalan numbers** 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...