

# MATH 110: Old final exam

Justify all solutions fully and show your work.

1. Write the (interesting) number 163 in the following numeral systems:

- (a) Base 2 (binary)

Successively subtracting powers of 2 gives

$$163 = 128 + 32 + 2 + 1 = 2^7 + 2^5 + 2^1 + 2^0.$$

So,

$$163_{10} = 10100011_2.$$

- (b) Base 6 (senary) *Remark: This numeral system is used by the Kanum- and Ndom-speaking peoples of New Guinea.*

Since  $6^2 = 36 \leq 163 < 216 = 6^3$ , the senary representation of 163 has 3 digits. First, dividing by  $6^2 = 36$  gives

$$163 = 4 \cdot 36 + 19.$$

Similarly, dividing the remainder 19 by 6 gives  $19 = 3 \cdot 6 + 1$ , yielding

$$163_{10} = 431_6.$$

- (c) Base 16 (hexadecimal)

Since  $16^1 = 16 \leq 163 < 16^2 = 256$ , the hexadecimal representation of 163 has 2 digits. Dividing by 16 gives

$$163 = 10 \cdot 16 + 3,$$

so

$$163_{10} = A3_{16}.$$

2. Recall that there are 23 students in our class.

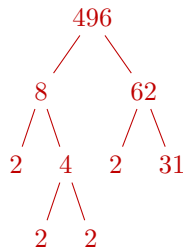
- (a) Suppose that each person in our class lists the following 3 types of ice cream from favorite to least favorite (without ties): chocolate, strawberry, vanilla. Can we conclude that at least two students will have the same list of preferences?

There are  $3! = 3 \cdot 2 \cdot 1 = 6$  possible lists of preferences. Since there are more students (23) than possible lists, the Pigeonhole Principle ensures that at least 2 students have the same list.

- (b) If we instead ask each person in our class to rank 4 types of ice cream—suppose we add pistachio as an option—can we conclude that at least 2 students will have the same list of preferences?

With a fourth flavor there are  $4! = 4 \cdot 3 \cdot 2 \cdot 1$  possible lists of preferences. Since there are more possible lists than students, it need not be the case that 2 students have the same list.

3. Use a tree diagram to find the prime factorization of 496.



So, the prime factorization of 496 is  $2^4 \cdot 31$ .

Alternatively, recall that 496 is the third perfect number, corresponding to  $p = 5$ , so  $496 = 2^{5-1}(2^5 - 1) = 2^4 \cdot 31$ .

4. Consider a set  $X$  with 4 elements.

(a) How many distinct functions  $X \rightarrow X$  are there?

Each of the 4 elements can be mapped to any of the 4 elements, so there  $4 \cdot 4 \cdot 4 \cdot 4 = 4^4 = \boxed{256}$  functions.

(b) How many of the functions  $X \rightarrow X$  are bijective?

We count the number of choices available for each value of a bijective function  $f : X \rightarrow X$ , and for convenience we denote the elements of the set by 1, 2, 3, 4: First,  $f(1)$  can take on any value,  $f(2)$  any value except  $f(1)$ ,  $f(3)$  either value not  $f(1)$  or  $f(2)$ , and  $f(4)$  to the value not among  $f(1), f(2), f(3)$ . So, there are  $4 \cdot 3 \cdot 2 \cdot 1 = 4! = \boxed{24}$  bijective functions  $X \rightarrow X$ .

(c) A function  $f : X \rightarrow X$  is an **involution** if  $(f \circ f)(x) = x$ . How many functions  $X \rightarrow X$  are involutions?

For an involution  $f$ , if there is some element  $a$  such  $b := f(a) \neq a$ . Then,  $f$  satisfies  $f(b) = a$ . So, for each value in  $X$  either  $f$  maps  $a$  to itself or it exchanges  $a$  with another element of  $X$ . Since  $X$  has 4 elements,  $f$  exchanges either 0 pairs, 1 pairs, or 2 pairs.

- If  $f$  exchanges 0 pairs, then  $f(x) = x$  for all  $x \in X$ , that is,  $f$  is the identity function  $\text{id}_X$ .
- If  $f$  exchanges 1 pairs, then there are  $\binom{4}{2} = 6$  possibilities for the pair of exchanged values. *Remark: Recall that these functions are, when regarded as permutations, the transpositions of elements  $X$ .*
- If  $f$  exchanges 2 pairs, then there are again  $\binom{4}{2} = 6$  possibilities for first pair of exchanged values; any such pair determines the second pair (just the two elements not in the first pair), and the order of pairs doesn't matter, so there are  $\frac{6}{2!} = 3$  possible functions. *Remark: Regarded as permutations, these functions are called the double transpositions of elements of  $X$ .*

In total there are  $6 + 3 + 1 = \boxed{10}$  involutions.

5. For each of following binary operations, (i) identify (and justify) whether it is commutative, and (ii) identify (and justify) whether it is associative.

(a) **(The averaging operation)**  $(\mathbb{R}, *)$ , where  $a * b = \frac{1}{2}(a + b)$

Using commutativity of real addition we have  $a * b = \frac{1}{2}(a + b) = \frac{1}{2}(b + a) = b * a$ , so  $\boxed{* \text{ is commutative }}$ .

On the other hand,  $(0 * 0) * 1 = 0 * 1 = \frac{1}{2}$  but  $0 * (0 * 1) = 0 * \frac{1}{2} = \frac{1}{4}$ , so  $\boxed{* \text{ is not associative }}$ .

(b)  $(X, \star)$ , where  $X$  is any set (with  $|X| > 1$ ) and  $a \star b = a$

Since  $|X| > 1$  there are distinct elements  $a, b \in X$ . Then,  $a \star b = a$  but  $b \star a = b \neq a$ , so  $\boxed{\star \text{ is not commutative }}$ .

On the other hand, for any  $a, b, c \in X$ , we have  $(a \star b) \star c = a \star c = a = a \star b = a \star (b \star c)$ , so  $\boxed{\star \text{ is associative }}$ .

6. In a few sentences, identify and explain some considerations when choosing a map projection to display geographical information. Give some examples of specific applications and projections appropriate for them.

(Many answers possible.)