CONFORMAL TRACTOR GEOMETRY COMPUTATIONS

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1. The standard representation $\mathbb{V} := \mathbb{R}^{p+1,q+1}$

A choice of scale determines a G_0-module decomposition of the standard representation $\mathbb{V}$, namely

$$\mathbb{V} \cong_{G_0} \mathbb{R}[1] \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}[-1].$$

1.1. The operators determined by a choice of scale.

Recall the operators

$$Y^A \in \mathbb{V}[-1], \qquad Z^A{}_a \in \mathbb{V}[1] \otimes (\mathbb{R}^{p,q})^*, \qquad X^A \in \mathbb{V}[1],$$

that inject the irreducible G_0-modules $\mathbb{W}$ into $\mathbb{V}$, and by equivariance, the operators that inject the respective associated bundles in the standard tractor bundle. Henceforth we often suppress the distinction between the algebraic and bundle settings. Only X^A is independent of the choice of scale.

The only nonzero contractions of these operators with the canonical nondegenerate, symmetric bilinear form $H_{AB} \in S^2\mathbb{V}^*$ are determined by

$$H_{AB} Y^A X^B = 1$$
$$H_{AB} Z^A{}_a Z^B{}_b = \mathbf{g}_{ab},$$

where $\mathbf{g}_{ab} \in S^2(\mathbb{R}^{p,q})^*[2]$ is the canonical conformal structure.

We can extract the components of the general standard tractor

$$I^A := \sigma Y^A + \mu^a Z^A{}_a + \rho X^A$$

via

$$\sigma = X_A I^A$$
$$\mu^a = Z_A{}^a I^A$$
$$\rho = Y_A I^A.$$

1.2. The normal standard tractor connection.

The covariant derivatives of these operators are given by

$$\nabla_c Y^A = P_{cd} Z^{Ad}$$
$$\nabla_c Z^A{}_b = -P_{bc} X^A - \mathbf{g}_{bc} Y^A$$
$$\nabla_c X^A = Z^A{}_c.$$

Thus, in terms of its components, the covariant derivative of a general standard tractor is

$$\nabla_b I^A = (\sigma_{,b} - \mu_b) Y^A + (\mu^a{}_{,b} + P^a{}_b \sigma + \delta^a{}_b \rho) Z^A{}_a + (\rho_{,b} - P_{ab}\mu^a) X^A.$$

2. The adjoint representation $\mathfrak{g} := \mathfrak{so}(\mathbb{V}) \cong \mathfrak{so}(p+1, q+1)$

A choice of scale σ determines a G_0-module decomposition of the adjoint representation $\mathfrak{so}(\mathbb{V})$, namely,

$$\mathfrak{so}(\mathbb{V}) \cong_{G_0} \mathbb{R}^{p,q} \oplus \begin{pmatrix} \mathfrak{so}(\mathbb{R}^{p,q}) \\ \oplus \\ \mathbb{R} \end{pmatrix} \oplus (\mathbb{R}^{p,q})^*$$

(Here, $\mathfrak{so}(\mathbf{g}) \cong \mathfrak{so}(p, q)$.)

2.1. **The operators determined by a choice of scale.** The usual operators $X^A, Z^A{}_b, Y^A$ determine corresponding operators that inject the irreducible G_0-modules $\mathbb{W}$ into $\mathfrak{so}(p+1,q+1)$, and by equivariance, operators that inject the respective associated bundles into the adjoint tractor bundle, $\mathcal{A}$:

$$\begin{aligned} Q^A{}_{Bd} &:= Y^A Z_{Bd} - Z^A{}_d Y_B && \in \mathfrak{so}(\mathbb{V}) \otimes (\mathbb{R}^{p,q})^* \\ R^A{}_{Bc}{}^d &:= \tfrac{1}{2}\left(Z^A{}_c Z_B{}^d - Z^{Ad} Z_{Bc}\right) && \in \mathfrak{so}(\mathbb{V}) \otimes \mathfrak{so}(\mathbb{R}^{p,q})^* \\ S^A{}_B &:= Y^A X_B - X^A Y_B && \in \mathfrak{so}(\mathbb{V}) \\ T^A{}_B{}^d &:= X^A Z_B{}^d - Z^{Ad} X_B && \in \mathfrak{so}(\mathbb{V}) \otimes \mathbb{R}^{p,q}. \end{aligned}$$

(Recall that via the musical isomorphisms $\mathbb{V} \cong \mathbb{V}^*$ and $\mathbb{R}^{p,q} \cong (\mathbb{R}^{p,q})^*[2]$, we have $\mathfrak{so}(p+1,q+1) \cong \Lambda^2\mathbb{V}^*$ and $\mathfrak{so}(p,q) \cong \Lambda^2(\mathbb{R}^{p,q})^*[2]$.) If we fix a basis for TM, then we can view these operators together as basis of $\mathfrak{so}(p+1,q+1)$ (or a frame for $\mathcal{A}$), where the indices vary over the chosen basis of TM and its dual basis as appropriate. Only $T^A{}_B{}^d$ is independent of the choice of scale.

Regarded as endomorphisms $U^A{}_B$ of the standard tractor bundle $\mathcal{T}$ indexed by Latin miniscules, the compositions of these operators are given by

$$\begin{aligned} Q^A{}_{Ce} Q^C{}_{Bg} &= -\mathbf{g}_{eg} Y^A Y_B \\ Q^A{}_{Ce} R^C{}_{Bf}{}^g &= \tfrac{1}{2}\left(\mathbf{g}_{ef} Y^A Z_B{}^g - \delta^g{}_e Y^A Z_{Bf}\right) \\ Q^A{}_{Ce} S^C{}_B &= Z^A{}_e Y_B \\ Q^A{}_{Ce} T^C{}_B{}^g &= -\delta^g{}_e Y^A X_B - Z^A{}_e Z_B{}^g \\ R^A{}_{Cd}{}^e Q^C{}_{Bg} &= \tfrac{1}{2}\left(-\delta^e{}_g Z^A{}_d Y_B + \mathbf{g}_{dg} Z^{Ae} Y_B\right) \\ R^A{}_{Cd}{}^e R^C{}_{Bf}{}^g &= \tfrac{1}{4}\left(-\mathbf{g}^{eg} Z^A{}_d Z_{Bf} + \delta^e{}_f Z^A{}_d Z_B{}^g + \delta^g{}_d Z^{Ae} Z_{Bf} - \mathbf{g}_{df} Z^{Ae} Z_B{}^g\right) \\ R^A{}_{Cd}{}^e S^C{}_B &= 0 \\ R^A{}_{Cd}{}^e T^C{}_B{}^g &= \tfrac{1}{2}\left(-\mathbf{g}^{eg} Z^A{}_d X_B + \delta^g{}_d Z^{Ae} X_B\right) \\ S^A{}_C Q^C{}_{Bg} &= Y^A Z_{Bg} \\ S^A{}_C R^C{}_{Bf}{}^g &= 0 \\ S^A{}_C S^C{}_B &= Y^A X_B + X^A Y_B \\ S^A{}_C T^C{}_B{}^g &= -X^A Z_B{}^g \\ T^A{}_C{}^e Q^C{}_{Bg} &= -\delta^e{}_g X^A Y_B - Z^{Ae} Z_{Bg} \\ T^A{}_C{}^e R^C{}_{Bf}{}^g &= \tfrac{1}{2}\left(-\mathbf{g}^{eg} X^A Z_{Bf} + \delta^e{}_f X^A Z_B{}^g\right) \\ T^A{}_C{}^e S^C{}_B &= -Z^{Ae} X_B \\ T^A{}_C{}^e T^C{}_B{}^g &= -\mathbf{g}^{eg} X^A X_B. \end{aligned}$$

The nonzero traces (in the majuscule indices) of these compositions are

$$\begin{aligned} Q^A{}_{Be} T^B{}_A{}^g &= -2\delta^g{}_e \\ R^A{}_{Bd}{}^e R^B{}_{Af}{}^g &= \tfrac{1}{2}\left(\delta^e{}_f \delta^g{}_d - \mathbf{g}^{eg}\mathbf{g}_{df}\right) \\ S^A{}_B S^B{}_A &= 2. \end{aligned}$$

So we can extract the components of a general adjoint tractor

$$\mathbb{A}^A{}_B := k^a Q^A{}_{Ba} + \varphi^a{}_b R^A{}_{Ba}{}^b + \xi S^A{}_B + \nu_b T^A{}_B{}^b$$

via

$$\begin{aligned} k^a &= -\tfrac{1}{2} T^B{}_A{}^a \mathbb{A}^A{}_B &&= X_A Z^{Ba} \mathbb{A}^A{}_B \\ \varphi^a{}_b &= R^B{}_{Ab}{}^a \mathbb{A}^A{}_B &&= Z_A{}^a Z^B{}_b \mathbb{A}^A{}_B \\ \xi &= \tfrac{1}{2} S^B{}_A \mathbb{A}^A{}_B &&= X_A Y^B \mathbb{A}^A{}_B \\ \nu_b &= -\tfrac{1}{2} Q^B{}_{Ab} \mathbb{A}^A{}_B &&= Y_A Z^B{}_b \mathbb{A}^A{}_B. \end{aligned}$$

The Lie brackets $[U,V]^A{}_B := U^A{}_C V^C{}_B - V^A{}_C U^C{}_B$ of the operators are given by

$$\begin{aligned}
[Q_e, Q_g]^A{}_B &= 0\\
[Q_e, R_f{}^g]^A{}_B &= \tfrac{1}{2}\left(-\delta^g{}_e Q^A{}_{Bf} + \mathbf{g}_{ef} Q^A{}_B{}^g\right)\\
[Q_e, S]^A{}_B &= -Q^A{}_{Be}\\
[Q_e, T^g]^A{}_B &= -2R^A{}_{Be}{}^g - \delta^g{}_e S^A{}_B\\
[R_d{}^e, R_f{}^g]^A{}_B &= \tfrac{1}{2}\left(\delta^e{}_f R^A{}_{Bd}{}^g - \mathbf{g}^{eg} R^A{}_{Bdf} - \mathbf{g}_{df} R^A{}_B{}^{eg} - \delta^g{}_d R^A{}_{Bf}{}^e\right)\\
[R_d{}^e, S]^A{}_B &= 0\\
[R_d{}^e, T^g]^A{}_B &= \tfrac{1}{2}\left(-\delta^g{}_d T^A{}_B{}^e + \mathbf{g}^{eg} T^A{}_{Bd}\right)\\
[S,S]^A{}_B &= 0\\
[S, T^g]^A{}_B &= -T^A{}_B{}^g\\
[T^e, T^g]^A{}_B &= 0.
\end{aligned}$$

2.2. **The Kostant codifferential.** The relevant Kostant codifferential in degree 1 is the map

$$\partial^* : \mathfrak{p}_+ \otimes \mathfrak{g} \to \mathfrak{g}$$

given by (the restriction of) the adjoint action. We can identify an element $\eta_c \in \mathfrak{p}_+$ with $\eta_c T^A{}_B{}^c \in \mathfrak{g}$, and hence ∂^* is given on simple elements by

$$\partial^*(\eta \otimes \mathbb{A})^D{}_E = [\eta_c T^c, \mathbb{A}]^D{}_E.$$

We wish to identify $\partial^*(\eta\otimes\mathbb{A})^D{}_E$ as a contraction of $(\eta\otimes\mathbb{A})_c{}^A{}_B = \eta_c \mathbb{A}^A{}_B$ with a natural tractor tensor, which we denote $(\partial^*)^A{}_B{}^c{}_D{}^E$, in $\mathfrak{g}\otimes(\mathfrak{p}_+\otimes\mathfrak{g})^* \cong \mathfrak{g}\otimes(\mathfrak{g}/\mathfrak{p})\otimes\mathfrak{g}^*$.

By linearity this will characterize ∂^* without any restriction to simple elements. Using the extraction formulae gives that we can write a general adjoint tractor as

$$\mathbb{A}^D{}_E = -\tfrac{1}{2} T^B{}_A{}^d \mathbb{A}^A{}_B Q^D{}_{Ed} + R^B{}_{Ae}{}^d \mathbb{A}^A{}_B R^D{}_{Ed}{}^e + \tfrac{1}{2} S^B{}_A \mathbb{A}^A{}_B S^D{}_E - \tfrac{1}{2} Q^B{}_{Ae} \mathbb{A}^A{}_B T^D{}_E{}^e.$$

So, substituting in the above formula for $\partial^*(\eta\otimes\mathbb{A})^D{}_E$, using linearity, factoring out the common factor $\eta_c\mathbb{A}^A{}_B$, and using the above formulae for Lie brackets of injection operators gives

$$\begin{aligned}
\partial^*(\eta\otimes\mathbb{A})^D{}_E &= [\eta_c T^c, \mathbb{A}]^D{}_E\\
&= \eta_c \mathbb{A}^A{}_B\left(-\tfrac{1}{2} T^B{}_A{}^d [T^c, Q_d]^D{}_E + R^B{}_{Ae}{}^d [T^c, R_d{}^e]^D{}_E + \tfrac{1}{2} S^B{}_A [T^c, S]^D{}_E - \tfrac{1}{2} Q^B{}_{Ae}[T^c, T^e]\right)\\
&= \eta_c \mathbb{A}^A{}_B\left[-\tfrac{1}{2} T^B{}_A{}^d(-2R^D{}_{Ed}{}^c + \delta^c{}_d S^D{}_E) + R^B{}_{Ae}{}^d\left[\tfrac{1}{2}(\delta^c{}_d T^D{}_E{}^e - \mathbf{g}^{ce} T^D{}_{Ed})\right] - \tfrac{1}{2} S^B{}_A(-T^D{}_E{}^c)\right]. ***
\end{aligned}$$

So, the operator with whose contraction characterizes the action of ∂^* is the one in the outer brackets on the right-hand side. Exploiting the fact that $R^B{}_{Ae}{}^d$ is $\mathbf{g}$-skew in its miniscule indices simplifies the middle term, giving a simple expression for the codifferential so viewed:

$$(\partial^*)_A{}^{BcD}{}_E = R^B{}_{Af}{}^c T^D{}_E{}^f + \tfrac{1}{2} S^B{}_A T^D{}_E{}^c - T^B{}_A{}^f R^D{}_{Ef}{}^c - \tfrac{1}{2} T^B{}_A{}^c S^D{}_E.$$

2.3. **The normal adjoint tractor connection.** Using the usual formulae for the normal tractor covariant derivatives of $X^A, Z^A{}_b, Y^A$ gives formulas for the derivatives $\nabla_e U^A{}_B$:

$$\begin{aligned}
\nabla_e Q^A{}_{Bd} &= -2P_{ef} R^A{}_{Bd}{}^f - P_{de} S^A{}_B\\
\nabla_e R^A{}_{Bc}{}^d &= \tfrac{1}{2}\left(\delta^d{}_e Q^A{}_{Bc} - \mathbf{g}_{ce} Q^A{}_B{}^d + P^d{}_e T^A{}_{Bc} - P_{ce} T^A{}_B{}^d\right)\\
\nabla_e S^A{}_B &= Q^A{}_{Be} - P_{ef} T^A{}_B{}^f\\
\nabla_e T^A{}_B{}^d &= 2R^A{}_{Be}{}^d + \delta^d{}_e S^A{}_B.
\end{aligned}$$

Thus, in terms of its components, the covariant derivative of a general adjoint tractor is

$$\begin{aligned}
\nabla_c \mathbb{A}^A{}_B &= \nabla_c(k^b Q^A{}_{Bb} + \varphi^a{}_b R^A{}_{Ba}{}^b - \xi S^A{}_B + \nu_b T^A{}_B{}^b)\\
&= (k^b{}_{,c} + \varphi^b{}_c + \xi\delta^b{}_c) Q^A{}_{Bb} + [\varphi^a{}_{b,c} + (P^a{}_c k_b - P_{bc} k^a) + (\delta^a{}_c \nu_b - g_{bc}\nu^a)] R^A{}_{Ba}{}^b\\
&\quad + (\xi_{,c} - P_{cd} k^d + \nu_c) S^A{}_B + (\nu_{b,c} - P_{cd}\varphi^d{}_b - \xi P_{bc}) T^A{}_B{}^b.
\end{aligned}$$

For convenience, for a given adjoint tractor $\mathbb{A}^A{}_B$ we denote the components of the covariant derivative $\nabla_c \mathbb{A}^A{}_B$ by

$$\begin{aligned}\alpha^b{}_c &:= k^b{}_{,c} + \varphi^b{}_c + \xi\delta^b{}_c \\ \beta^a{}_{bc} &:= \varphi^a{}_{b,c} + (P^a{}_c k_b - P_{bc}k^a) + (\delta^a{}_c \nu_b - \mathbf{g}_{bc}\nu^a) \\ \gamma_c &:= \xi_{,c} - P_{cd}k^d + \nu_c \\ \zeta_{bc} &:= \nu_{b,c} - P_{cd}\varphi^d{}_b - \xi P_{bc}\end{aligned}$$

so that

$$\nabla_c \mathbb{A}^A{}_B := \alpha^a{}_c Q^A{}_{Ba} + \beta^a{}_{bc} R^A{}_{Ba}{}^b + \gamma_c S^A{}_B + \zeta_{bc} T^A{}_B{}^b.$$

2.4. **The adjoint splitting operator.** The adjoint splitting operator $L_0 : \Gamma(TM) \to \Gamma(\mathcal{A})$ for the normal tractor connection ∇ is characterized by (1) $\Pi_0 \circ L_0 = \mathrm{id}_{TM}$, where $\Pi_0 : \mathcal{A} \to TM$ is the canonical projection induced by the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$, which we can identify with $T^A{}_B{}^c$ (also, we freely use the same notation for tensorial maps and the maps on spaces of sections they induce), and (2) $\partial^* \circ \nabla \circ L_0 = 0$.

To determine the components of this operator with respect to the splitting determined by a choice of scale, we compute

$$(\partial^* \nabla \mathbb{A})^D{}_E$$

for a general adjoint tractor $\mathbb{A}$ as above. Expanding gives

$$\begin{aligned}(\partial^* \nabla \mathbb{A})^D{}_E &= (\partial^*)_A{}^{BcD}{}_E \nabla_c \mathbb{A}^A{}_B \\ &= \left(R^B{}_{Af}{}^c T^D{}_E{}^f + \tfrac{1}{2} S^B{}_A T^D{}_E{}^c - T^B{}_A{}^f R^D{}_{Ef}{}^c - \tfrac{1}{2} T^B{}_A{}^c S^D{}_E\right) \\ &\qquad \cdot (\alpha^a{}_c Q^A{}_{Ba} + \beta^a{}_{bc} R^A{}_{Ba}{}^b + \gamma_c S^A{}_B + \zeta_{bc} T^A{}_B{}^b) \\ &= -2\alpha^a{}_c R^D{}_{Ea}{}^c + \alpha^a{}_a S^D{}_E + \tfrac{1}{2}(\beta^a{}_{ca} - \beta_c{}^a{}_a + \gamma_c) T^D{}_E{}^c.\end{aligned}$$

We now impose the condition $\partial^* \nabla \mathbb{A} = 0$. Contracting with $R^E{}_{Df}{}^g$ and substituting the expression for α in terms of the components of $\mathbb{A}$ gives

$$\begin{aligned}0 &= -(\alpha^g{}_f - \alpha_g{}^f) \\ &= -[(k^g{}_{,f} - \varphi^g{}_f + \xi\delta^g{}_f) - (k_{f,}{}^g - \varphi_f{}^g + \xi\delta_f{}^g)] \\ &= -[k^g{}_{,f} - k_{f,}{}^g - 2\varphi^g{}_f].\end{aligned}$$

Solving for φ gives

$$\varphi^g{}_f = \tfrac{1}{2}(-k^g{}_{,f} + k_{f,}{}^g).$$

Instead contracting with $S^E{}_D$ gives

$$\begin{aligned}0 &= 2\alpha^a{}_a \\ &= 2(k^a{}_{,a} + \varphi^a{}_a + \xi\delta^a{}_a) \\ &= 2(k^a{}_{,a} + n\xi),\end{aligned}$$

and again solving gives

$$\xi = -\tfrac{1}{n} k^a{}_{,a}.$$

Finally, contracting instead with $Q^E{}_{Df}$ gives

$$\begin{aligned}0 &= 2(-\beta^a{}_{fa} + \gamma_f) \\ &= 2[(\varphi^a{}_{f,a} + (P^a{}_a k_f - P^a{}_g k_a) + (\delta^a{}_a \nu_f - \delta^a{}_f \nu_a)) + 2(\xi_{,f} + \nu_g - P_{fg}k^g)] \\ &= 2[\varphi^a{}_{f,a} + 2P^a{}_a k_f - 4P^a{}_g k_a + 2\xi_{,f} + 2n\nu_f].\end{aligned}$$

Solving for ν and substituting the already-derived expressions for φ and ξ gives

$$\begin{aligned}\nu_f &= \tfrac{1}{n}\left(-\varphi^a{}_{f,a} - P^a{}_a k_f + 2P_{af}k^a - \xi_{,f}\right) \\ &= \tfrac{1}{n}\left(-\tfrac{1}{2}(-k^a{}_{,f} + k_{f,}{}^a)_{,a} - P^a{}_a k_f + 2P_{af}k^a - \left(-\tfrac{1}{n}k^a{}_{,a}\right)_{,g}\right) \\ &= \tfrac{1}{n}\left(\tfrac{1}{2}k^a{}_{,fa} - \tfrac{1}{2}k_{f,}{}^a{}_a + \tfrac{1}{n}k^a{}_{,af} - P^a{}_a k_f + 2P_{af}k^a\right)\end{aligned}$$

We have thus determined the normal splitting operator $L_0 : \Gamma(TM) \to \Gamma(\mathcal{A})$ for the adjoint representation:

$$L_0(k)^A{}_B = k^a Q^A{}_{Ba} + \tfrac{1}{2}(-k^a{}_{,b} + k_{b,}{}^a) R^A{}_{Ba}{}^b - \tfrac{1}{n}k^c{}_{,c} S^A{}_B + \tfrac{1}{n}\left(-\tfrac{1}{2}k_{b,}{}^c{}_c + \tfrac{1}{2}k^c{}_{,bc} + \tfrac{1}{n}k^c{}_{,cb} - P^c{}_c k_b + 2P_{bc}k^c\right) T^A{}_B{}^b.$$

3. The standard representation $\mathbb{V}$, continued

3.1. **The Kostant codifferential.** The relevant Kostant codifferential in degree 1 is the map $\partial^* : \mathfrak{p}_+ \otimes \mathbb{V} \to \mathbb{V}$ given by (the restriction of) the standard action. This is given on simple elements by $\partial^*(\eta \otimes I)^C = [(\eta_b T^b) \cdot I]^C$, where $\cdot$ denotes the standard action. Computing gives that ∂^* is given by the contraction with the following tractor tensor (which we also denote ∂^*):

$$(\partial^*)_A{}^{bC} = T^C{}_A{}^b = X^C Z_A{}^b - Z^{Cb} X_A.$$

3.2. **The standard splitting operator.** Solving $(\partial^* \nabla I)^A = 0$ gives that the normal splitting operator $L_0 : \mathcal{E}[1] \to \mathcal{T}$ for the standard representation is

$$L_0(\sigma)^A := \sigma Y^A + \sigma^{,b} Z^A{}_b - \tfrac{1}{n}(\sigma_{,b}{}^b + P^b{}_b \sigma) X^A.$$

4. The kth alternating representation, $\Lambda^k \mathbb{V}^*$

4.1. **The operators determined by a choice of scale.** The usual operators $X^A, Z^A{}_b, Y^A$ determine corresponding operators that inject the irreducible G_0-modules $\mathbb{W}$ into $\Lambda^k \mathbb{V}$. Here $\mathbf{A}$ is an abbreviation for the multi-index $A_1 \cdots A_k$ and $\mathbf{b}$ is an abbreviation for the multi-index $b_1 \cdots b_k$. The notation $\dot{\mathbf{A}}$ is an abbreviation for the multi-index $A_2 \cdots A_k$ and $\ddot{\mathbf{A}}$ an abbreviation for $A_3 \cdots A_k$; define analogously $\dot{\mathbf{b}}$ and $\ddot{\mathbf{b}}$, denote by $\dddot{\mathbf{b}}$ the multi-index $b_4 \cdots b_k$, and set $Z_{\mathbf{A}}{}^{\mathbf{b}} := Z_{[A_1}{}^{b_1} \cdots Z_{A_k]}{}^{b_k}$. (Recall that the musical isomorphism $\mathbb{V}^* \cong \mathbb{V}$ determined by H induces an isomorphism $\Lambda^2 \mathbb{V}^* \cong \mathfrak{so}(p+1, q+1)$. The notation in this section is chosen to be compatible with the notation in the above section giving results for the adjoint representation.)

$$\begin{aligned}
Q_{\mathbf{A}}{}^{\dot{\mathbf{b}}} &:= k Y_{[A_1} Z_{\dot{\mathbf{A}}]}{}^{\dot{\mathbf{b}}} && \in \Lambda^k \mathbb{V}^* \otimes \Lambda^{k-1} \mathbb{R}^{p,q}[-k] \\
R_{\mathbf{A}}{}^{\mathbf{b}} &:= Z_{\mathbf{A}}{}^{\mathbf{b}} && \in \Lambda^k \mathbb{V}^* \otimes \Lambda^k \mathbb{R}^{p,q}[-k] \\
S_{\mathbf{A}}{}^{\ddot{\mathbf{b}}} &:= k(k-1) Y_{[A_1} X_{A_2} Z_{\ddot{\mathbf{A}}]}{}^{\ddot{\mathbf{b}}} && \in \Lambda^k \mathbb{V}^* \otimes \Lambda^{k-2} \mathbb{R}^{p,q}[-k+2] \\
T_{\mathbf{A}}{}^{\dot{\mathbf{b}}} &:= k X_{[A_1} Z_{\dot{\mathbf{A}}]}{}^{\dot{\mathbf{b}}} && \in \Lambda^k \mathbb{V}^* \otimes \Lambda^{k-1} \mathbb{R}^{p,q}[-k+2].
\end{aligned}$$

If we denote the operators defined for a given k by $(Q_k)_{\mathbf{A}}{}^{\dot{\mathbf{b}}}$ (and similarly for the other operators), then the operators for various k are related by the special cases

$$(Q_1)_A = Y_A, \qquad (R_1)_A{}^b = Z_A{}^b, \qquad (T_1)_A = X_A,$$

and (for appropriate k) the contraction identities

$$\begin{array}{c|ccc}
 & Y^{A_1} & Z^{A_1}{}_c & X^{A_1} \\
\hline
(Q_k)_{A_1 \dot{\mathbf{A}}}{}^{\dot{\mathbf{b}}} & 0 & -\delta^{[b_2}{}_c (Q_{k-1})_{\dot{\mathbf{A}}}{}^{\ddot{\mathbf{b}}]} & (R_{k-1})_{\dot{\mathbf{A}}}{}^{\dot{\mathbf{b}}} \\
(R_k)_{A_1 \dot{\mathbf{A}}}{}^{\mathbf{b}} & 0 & \delta^{[b_1}{}_c (R_{k-1})_{\dot{\mathbf{A}}}{}^{\dot{\mathbf{b}}]} & 0 \\
(S_k)_{A_1 \mathbf{A}}{}^{\ddot{\mathbf{b}}} & -(Q_{k-1})_{\dot{\mathbf{A}}}{}^{\ddot{\mathbf{b}}} & \delta^{[b_3}{}_c (S_{k-1})_{\dot{\mathbf{A}}}{}^{\dddot{\mathbf{b}}]} & (T_{k-1})_{\dot{\mathbf{A}}}{}^{\ddot{\mathbf{b}}} \\
(T_k)_{A_1 \mathbf{A}}{}^{\dot{\mathbf{b}}} & (R_{k-1})_{\dot{\mathbf{A}}}{}^{\dot{\mathbf{b}}} & -\delta^{[b_2}{}_c (T_{k-1})_{\dot{\mathbf{A}}}{}^{\ddot{\mathbf{b}}]} & 0
\end{array}.$$

and the wedge product identities

$$\begin{array}{c|ccc}
 & Y_{A_0} & Z_{A_0}{}^c & X_{A_0} \\
\hline
(Q_k)_{\mathbf{A}}{}^{\dot{\mathbf{b}}} & 0 & -k(Q_{k+1})_{A_0 \mathbf{A}}{}^{c\dot{\mathbf{b}}} & -(S_{k+1})_{A_0 \mathbf{A}}{}^{\dot{\mathbf{b}}} \\
(R_k)_{\mathbf{A}}{}^{\mathbf{b}} & (Q_{k+1})_{A_0 \mathbf{A}}{}^{\mathbf{b}} & (k+1)(R_{k+1})_{A_0 \mathbf{A}}{}^{c\mathbf{b}} & (T_{k+1})_{A_0 \mathbf{A}}{}^{\mathbf{b}} \\
(S_k)_{\mathbf{A}}{}^{\ddot{\mathbf{b}}} & 0 & (k-1)(S_{k+1})_{A_0 \mathbf{A}}{}^{c\ddot{\mathbf{b}}} & 0 \\
(T_k)_{\mathbf{A}}{}^{\dot{\mathbf{b}}} & (S_{k+1})_{A_0 \mathbf{A}}{}^{\dot{\mathbf{b}}} & -k(T_{k+1})_{A_0 \mathbf{A}}{}^{c\dot{\mathbf{b}}} & 0
\end{array}.$$

denote, for example, $(Y \wedge (R_k)^{\mathbf{b}})_{A_0 \mathbf{A}} = (Q_{k+1})_{A_0 \mathbf{A}}{}^{\mathbf{b}}$.

The nonzero full traces on the tractor indices $\mathbf{A}$ of these operators are

$$\begin{aligned}
Q_{\mathbf{A}}{}^{\dot{\mathbf{b}}} T^{\mathbf{A}}{}_{\dot{\mathbf{c}}} &= k \delta^{[b_2}{}_{[c_2} \cdots \delta^{b_k]}{}_{c_k]} \\
R_{\mathbf{A}}{}^{b_1 \cdots b_k} R^{\mathbf{A}}{}_{c_1 \cdots c_k} &= \delta^{[b_1}{}_{[c_1} \cdots \delta^{b_k]}{}_{c_k]} \\
S_{\mathbf{A}}{}^{b_2 \cdots b_{k-1}} S^{\mathbf{A}}{}_{c_2 \cdots c_{k-1}} &= -k(k-1) \delta^{[b_2}{}_{[c_2} \cdots \delta^{b_{k-1}]}{}_{c_{k-1}]}.
\end{aligned}$$

So, we can extract the components of a general tractor k-form

$$\mathbb{B}_{\mathbf{A}} := \omega_{\dot{\mathbf{b}}} Q_{\mathbf{A}}{}^{\dot{\mathbf{b}}} + \phi_{\mathbf{b}} R_{\mathbf{A}}{}^{\mathbf{b}} + \mu_{\ddot{\mathbf{b}}} S_{\mathbf{A}}{}^{\ddot{\mathbf{b}}} + \tau_{\mathbf{b}} T_{\mathbf{A}}{}^{\dot{\mathbf{b}}}$$

via

$$\begin{aligned}
\omega_{\dot{\mathbf{b}}} &= \frac{1}{k} T^{\mathbf{A}}{}_{\mathbf{b}} \mathbb{B}_{\mathbf{A}} &&= X^{A_1} Z^{A_2}{}_{b_2} \cdots Z^{A_k}{}_{b_k} \mathbb{B}_{\mathbf{A}} \\
\phi_{\mathbf{b}} &= R^{\mathbf{A}}{}_{\mathbf{b}} \mathbb{B}_{\mathbf{A}} &&= Z^{A_1}{}_{b_1} \cdots Z^{A_k}{}_{b_k} \mathbb{B}_{\mathbf{A}} \\
\mu_{\ddot{\mathbf{b}}} &= -\frac{1}{k(k-1)} S^{\mathbf{A}}{}_{\ddot{\mathbf{b}}} \mathbb{B}_{\mathbf{A}} &&= X^{A_1} Y^{A_2} Z^{A_3}{}_{b_3} \cdots Z^{A_k}{}_{A_k} \mathbb{B}_{\mathbf{A}} \\
\tau_{\dot{\mathbf{b}}} &= \frac{1}{k} Q^{\mathbf{A}}{}_{\dot{\mathbf{b}}} \mathbb{B}_{\mathbf{A}} &&= Y^{A_1} Z^{A_2}{}_{b_2} \cdots Z^{A_k}{}_{b_k} \mathbb{B}_{\mathbf{A}}.
\end{aligned}$$

In particular, we can write the identity map $\delta_{\mathbf{A}}{}^{\mathbf{C}}$ on $\Lambda^k \mathbb{V}^*$ as

$$\delta_{\mathbf{A}}{}^{\mathbf{C}} = \frac{1}{k} Q_{\mathbf{A}}{}^{\dot{\mathbf{d}}} T^{\mathbf{C}}{}_{\dot{\mathbf{d}}} + R_{\mathbf{A}}{}^{\mathbf{d}} R^{\mathbf{C}}{}_{\mathbf{d}} - \frac{1}{k(k-1)} S_{\mathbf{A}}{}^{\ddot{\mathbf{d}}} S^{\mathbf{C}}{}_{\ddot{\mathbf{d}}} + \frac{1}{k} T_{\mathbf{A}}{}^{\dot{\mathbf{d}}} Q^{\mathbf{C}}{}_{\dot{\mathbf{d}}}.$$

The actions of T^c on the injectors are given by

$$\begin{aligned}
(T^c \cdot Q^{\dot{\mathbf{b}}})_{\mathbf{A}} &= -R_{\mathbf{A}}{}^{c\dot{\mathbf{b}}} + \frac{1}{k} \mathbf{g}^{c[b_2} S_{\mathbf{A}}{}^{\ddot{\mathbf{b}}]} \\
(T^c \cdot R^{\mathbf{b}})_{\mathbf{A}} &= \mathbf{g}^{c[b_1} T_{\mathbf{A}}{}^{\dot{\mathbf{b}}]} \\
(T^c \cdot S^{\ddot{\mathbf{b}}})_{\mathbf{A}} &= (k-1) T_{\mathbf{A}}{}^{c\ddot{\mathbf{b}}} \\
(T^c \cdot T^{\dot{\mathbf{b}}})_{\mathbf{A}} &= 0.
\end{aligned}$$

4.2. **The Kostant codifferential.** The relevant Kostant codifferential in degree 1 is the map

$$\partial^* : \mathfrak{p}_+ \otimes \Lambda^k \mathbb{V}^* \to \Lambda^k \mathbb{V}^*$$

given by (the restriction of) the standard action of $\mathfrak{g}$, which is given by the action of T^C on the injectors. As in the adjoint case, we wish to identify $\partial^*(\eta \otimes \mathbb{B})_{\mathbf{A}} = (\eta_c T^c \cdot \mathbb{B})_{\mathbf{A}}$ as a contraction of $(\eta \otimes \mathbb{B})_{b\mathbf{C}} = \eta_b \mathbb{B}_{\mathbf{C}}$ with a natural tractor tensor, which we denote $(\partial^*)_{\mathbf{A}}{}^{b\mathbf{C}}$, in $(\Lambda^k \mathbb{V}^*) \otimes (\mathfrak{p}_+ \otimes \Lambda^k \mathbb{V}^*)^* \cong \Lambda^k \mathbb{V}^* \otimes (\mathfrak{g}/\mathfrak{p}) \otimes \Lambda^k \mathbb{V}$. By linearity this will characterize ∂^*. Using the above formula for the identity $\delta^{\mathbf{C}}{}_{\mathbf{A}}$ on $\Lambda^k \mathbb{V}^*$ gives

$$\begin{aligned}
\mathbb{B}_{\mathbf{A}} &= \delta_{\mathbf{A}}{}^{\mathbf{C}} \mathbb{B}_{\mathbf{C}} \\
&= \left(\frac{1}{k} Q_{\mathbf{A}}{}^{\dot{\mathbf{d}}} T^{\mathbf{C}}{}_{\dot{\mathbf{d}}} + R_{\mathbf{A}}{}^{\mathbf{d}} R^{\mathbf{C}}{}_{\mathbf{d}} - \frac{1}{k(k-1)} S_{\mathbf{A}}{}^{\ddot{\mathbf{d}}} S^{\mathbf{C}}{}_{\ddot{\mathbf{d}}} + \frac{1}{k} T_{\mathbf{A}}{}^{\dot{\mathbf{d}}} Q^{\mathbf{C}}{}_{\dot{\mathbf{d}}} \right) \mathbb{B}_{\mathbf{C}}.
\end{aligned}$$

Substituting gives

$$\begin{aligned}
(\partial^*)_{\mathbf{A}}{}^{b\mathbf{C}} \eta_b \mathbb{B}_{\mathbf{C}} &= \partial^*(\eta \otimes \mathbb{B})_{\mathbf{A}} \\
&= \eta_b (T^b \cdot \mathbb{B})_{\mathbf{A}} \\
&= \eta_b \left(\frac{1}{k} (T^b \cdot Q^{\dot{\mathbf{d}}})_{\mathbf{A}} T^{\mathbf{C}}{}_{\dot{\mathbf{d}}} + (T^b \cdot R^{\mathbf{d}})_{\mathbf{A}} R^{\mathbf{C}}{}_{\mathbf{d}} - \frac{1}{k(k-1)} (T^b \cdot S^{\ddot{\mathbf{d}}})_{\mathbf{A}} S^{\mathbf{C}}{}_{\ddot{\mathbf{d}}} + \frac{1}{k} (T^b \cdot T^{\dot{\mathbf{d}}})_{\mathbf{A}} Q^{\mathbf{C}}{}_{\dot{\mathbf{d}}} \right) \mathbb{B}_{\mathbf{C}}.
\end{aligned}$$

So, using the above formulae for the actions of T^c on the injectors gives

$$(\partial^*)_{\mathbf{A}}{}^{b\mathbf{C}} = \left[\mathbf{g}^{c[d_1} T_{\mathbf{A}}{}^{\dot{\mathbf{d}}]} R^{\mathbf{C}}{}_{\mathbf{d}} - \frac{1}{k} T_{\mathbf{A}}{}^{c\ddot{\mathbf{d}}} S^{\mathbf{C}}{}_{\ddot{\mathbf{d}}} + \frac{1}{k} \left(-R_{\mathbf{A}}{}^{c\dot{\mathbf{d}}} + \frac{1}{k} \mathbf{g}^{c[d_2} S_{\mathbf{A}}{}^{\ddot{\mathbf{d}}]} \right) T^{\mathbf{C}}{}_{\dot{\mathbf{d}}} \right].$$

4.3. **The normal tractor k-form connection.** Using the usual formulae for the normal tractor covariant derivatives of $X^A, Z^A{}_b, Y^A$ gives formulas for the derivatives $\nabla_e U_{\mathbf{A}}$:

$$\begin{aligned}
\nabla_c Q_{\mathbf{A}}{}^{\dot{\mathbf{b}}} &= k P_{cd} R_{\mathbf{A}}{}^{d\dot{\mathbf{b}}} - P^{[b_2}{}_c S_{\mathbf{A}}{}^{\dot{\mathbf{b}}]} \\
\nabla_c R_{\mathbf{A}}{}^{\mathbf{b}} &= -\delta^{[b_1}{}_c Q_{\mathbf{A}}{}^{\dot{\mathbf{b}}]} - P_c{}^{[b_1} T_{\mathbf{A}}{}^{\dot{\mathbf{b}}]} \\
\nabla_c S_{\mathbf{A}}{}^{\ddot{\mathbf{b}}} &= (k-1) Q_{\mathbf{A}c}{}^{\ddot{\mathbf{b}}} - (k-1) P_{cd} T_{\mathbf{A}}{}^{d\ddot{\mathbf{b}}} \\
\nabla_c T_{\mathbf{A}}{}^{\dot{\mathbf{b}}} &= k R_{\mathbf{A}c}{}^{\dot{\mathbf{b}}} + \delta^{[b_2}{}_c S_{\mathbf{A}}{}^{\ddot{\mathbf{b}}]}.
\end{aligned}$$

Thus, in components, the covariant derivative of a general tractor k-form is

$$\begin{aligned}\nabla_b \mathbb{B}_{\mathbf{A}} &= \nabla_c(\omega_{\dot{\mathbf{b}}} Q_{\mathbf{A}}{}^{\dot{\mathbf{b}}} + \phi_{\mathbf{b}} R_{\mathbf{A}}{}^{\mathbf{b}} + \mu_{\ddot{\mathbf{b}}} S_{\mathbf{A}}{}^{\ddot{\mathbf{b}}} + \tau_{\dot{\mathbf{b}}} T_{\mathbf{A}}{}^{\dot{\mathbf{b}}}) \\ &= [\omega_{\dot{\mathbf{b}},c} - \phi_{c\dot{\mathbf{b}}} + (k-1) g_{c[b_2}\mu_{\dot{\mathbf{b}}]}] Q_{\mathbf{A}}{}^{\dot{\mathbf{b}}} + (\phi_{\mathbf{b},c} + kP_{c[b_1}\omega_{\dot{\mathbf{b}}]} + k\mathbf{g}_{c[b_1}\tau_{\dot{\mathbf{b}}]}) R^{\mathbf{b}}_{\mathbf{A}} \\ &\qquad + (\mu_{\ddot{\mathbf{b}},c} - P_c{}^d \omega_{d\ddot{\mathbf{b}}} + \tau_{c\ddot{\mathbf{b}}}) S_{\mathbf{A}}{}^{\ddot{\mathbf{b}}} + (\tau_{\dot{\mathbf{b}},c} - P_c{}^d \phi_{d\dot{\mathbf{b}}} - (k-1) P_{c[b_2}\mu_{\dot{\mathbf{b}}]}) T_{\mathbf{A}}{}^{\dot{\mathbf{b}}}.\end{aligned}$$

Again for convenience, for a given adjoint tractor $\mathbb{B}_{\mathbf{A}}$ we denote the components of the covariant derivative$\nabla_b \mathbb{B}_{\mathbf{A}}$ by

$$\begin{aligned}\alpha_{\dot{\mathbf{b}}c} &:= \omega_{\dot{\mathbf{b}},c} - \phi_{c\dot{\mathbf{b}}} + (k-1) g_{c[b_2}\mu_{\ddot{\mathbf{b}}]} \\ \beta_{\mathbf{b}c} &:= \phi_{\mathbf{b},c} + kP_{c[b_1}\omega_{\dot{\mathbf{b}}]} + k\mathbf{g}_{c[b_1}\tau_{\dot{\mathbf{b}}]} \\ \gamma_{\ddot{\mathbf{b}}c} &:= \mu_{\ddot{\mathbf{b}},c} - P_c{}^d \omega_{d\ddot{\mathbf{b}}} + \tau_{c\ddot{\mathbf{b}}} \\ \zeta_{\dot{\mathbf{b}}c} &:= \tau_{\dot{\mathbf{b}},c} - P_c{}^d \phi_{d\dot{\mathbf{b}}} - (k-1) P_{c[b_1}\mu_{\ddot{\mathbf{b}}]}\end{aligned}$$

so that

$$\nabla_c \mathbb{B}_{\mathbf{A}} := \alpha_{\dot{\mathbf{b}}c} Q_{\mathbf{A}}{}^{\dot{\mathbf{b}}} + \beta_{\mathbf{b}c} R_{\mathbf{A}}{}^{\mathbf{b}} + \gamma_{\ddot{\mathbf{b}}c} S_{\mathbf{A}}{}^{\ddot{\mathbf{b}}} + \zeta_{\dot{\mathbf{b}}c} T_{\mathbf{A}}{}^{\dot{\mathbf{b}}}.$$

4.4. **The splitting operator for $\Lambda^k \mathbb{V}^*$.** As in the adjoint case, to compute the splitting operator $L_0 : \Gamma(\Lambda^{k-1} T^* M[k]) \to \Gamma(\Lambda^k \mathcal{T}^*)$ with respect to the splitting determined by a choice of scale, we compute

$$(\partial^* \nabla \mathbb{B})_{\mathbf{A}}$$

for a general adjoint tractor $\mathbb{B}$. Expanding gives

$$\begin{aligned}(\partial^* \nabla \mathbb{B})_{\mathbf{A}} &= (\partial^*)_{\mathbf{A}}{}^{b\mathbf{C}} \nabla_b \mathbb{B}_{\mathbf{C}} \\ &= \left[\mathbf{g}^{b[d_1} T_{\mathbf{A}}{}^{\dot{\mathbf{d}}]} R^{\mathbf{C}}{}_{\mathbf{d}} - \frac{1}{k} T_{\mathbf{A}}{}^{b\ddot{\mathbf{d}}} S^{\mathbf{C}}{}_{\ddot{\mathbf{d}}} + \frac{1}{k}\left(-R_{\mathbf{A}}{}^{b\dot{\mathbf{d}}} + \frac{1}{k}\mathbf{g}^{b[d_2} S_{\mathbf{A}}{}^{\ddot{\mathbf{d}}]}\right) T^{\mathbf{C}}{}_{\dot{\mathbf{d}}}\right] \\ &\qquad \cdot (\alpha_{\dot{\mathbf{e}}b} Q_{\mathbf{C}}{}^{\dot{\mathbf{e}}} + \beta_{\mathbf{e}b} R_{\mathbf{C}}{}^{\mathbf{e}} + \gamma_{\ddot{\mathbf{e}}b} S_{\mathbf{C}}{}^{\ddot{\mathbf{e}}} + \zeta_{\dot{\mathbf{e}}b} T_{\mathbf{C}}{}^{\dot{\mathbf{e}}}) \\ &= -\alpha_{\dot{\mathbf{d}}b} R_{\mathbf{A}}{}^{b\dot{\mathbf{d}}} + \frac{1}{k}\mathbf{g}^{bd_2}\alpha_{\dot{\mathbf{d}}b} S_{\mathbf{A}}{}^{\ddot{\mathbf{d}}} + [\mathbf{g}^{bd_1}\beta_{\mathbf{d}b} + (k-1)\gamma_{\ddot{\mathbf{d}}d_2}] T_{\mathbf{A}}{}^{b\dot{\mathbf{d}}}.\end{aligned}$$

We now impose the condition $\partial^* \nabla \mathbb{B} = 0$. Contracting with $R^{\mathbf{A}}{}_{e_1 \cdots e_k}$ and substituting the expression for α in terms of the components of $\mathbb{B}$ gives

$$\begin{aligned}0 &= -\alpha_{[\mathbf{e}]} \\ &= -[\omega_{[\dot{\mathbf{e}},e_1]} - \phi_{\mathbf{e}} + (k-1)\mathbf{g}_{[e_1 e_2}\mu_{\ddot{\mathbf{e}}]}].\end{aligned}$$

The third term vanishes by symmetry, so rearranging gives

$$\phi_{\mathbf{e}} = \omega_{[\dot{\mathbf{e}},e_1]}.$$

Instead contracting with $S^{\mathbf{A}}{}_{\ddot{\mathbf{b}}}$ gives

$$\begin{aligned}0 &= \frac{1}{k}\mathbf{g}^{be_2}\alpha_{\dot{\mathbf{e}}b} \\ &= \frac{1}{k}\mathbf{g}^{be_2}[\omega_{[\dot{\mathbf{e}},b]} - \phi_{b\dot{\mathbf{e}}} + (k-1) g_{b[e_2}\mu_{\ddot{\mathbf{e}}]}].\end{aligned}$$

The second term in brackets vanishes by skew-symmetry, and we can expand the third term according to the position of the index e_2 as

$$(k-1)\mathbf{g}_{b[e_2}\mu_{\ddot{\mathbf{e}}]} = g_{be_2}\mu_{\ddot{\mathbf{e}}} - (k-2) g_{b[e_3}\mu_{|e_2|}{}^{\cdot}{}_{\ddot{\mathbf{e}}^{\cdot}]},$$

and the $\mathbf{g}^{be_2}$-trace of this term is just $(n-k+2)\mu_{\ddot{\mathbf{e}}}$, so substituting and rearranging gives

$$\mu_{\ddot{\mathbf{e}}} = -\frac{1}{n-k+2}\omega_{b\ddot{\mathbf{e}},}{}^{b}.$$

Finally, contracting instead with $Q^{\mathbf{A}}{}_{\dot{\mathbf{e}}}$ gives

$$\begin{aligned}0 &= k[\mathbf{g}^{bd}\beta_{d\dot{\mathbf{e}}b} + (k-1)\gamma_{\ddot{\mathbf{e}}e_2}] \\ &= k[\mathbf{g}^{bd}(\phi_{d\dot{\mathbf{e}},b} + kP_{b[d}\omega_{\dot{\mathbf{e}}]} + k\mathbf{g}_{b[d}\tau_{\dot{\mathbf{e}}]}) + (k-1)(\mu_{[\ddot{\mathbf{e}},e_2]} P^b{}_{[e_2}\omega_{|b|\ddot{\mathbf{e}}]} + \tau_{\dot{\mathbf{e}}})]\end{aligned}$$

Decomposing the second term according to the position of the index d gives

$$kP_{b[d}\omega_{\dot{\mathbf{e}}]} = P_{bd}\omega_{\dot{\mathbf{e}}} - (k-1) P^b{}_{[e_2}\omega_{|b|\ddot{\mathbf{e}}]},$$

and one can treat the third term separately. Carrying out the contraction with g^{bd} and collecting like terms then gives

$$0 = k[\phi_{b\dot{\mathbf{e}},}{}^{b} - 2(k-1)P^{b}{}_{[e_2}\omega_{|b|\ddot{\mathbf{e}}]} + P^{b}{}_{b}\omega_{\dot{\mathbf{e}}} + (k-1)\mu_{[\ddot{\mathbf{e}},e_2]} + n\tau_{\dot{\mathbf{e}}}],$$

and solving for $\tau_{\dot{\mathbf{e}}}$ yields

$$\tau_{\dot{\mathbf{e}}} = -\tfrac{1}{n}[\phi_{b\dot{\mathbf{e}},}{}^{b} - 2(k-1)P^{b}{}_{[e_2}\omega_{|b|\ddot{\mathbf{e}}]} + P^{b}{}_{b}\omega_{\dot{\mathbf{e}}} + (k-1)\mu_{[\ddot{\mathbf{e}},e_2]}].$$

Using the value of $\phi_{\mathbf{e}}$ determined above, the first term in brackets is

$$\phi_{b\dot{\mathbf{e}},}{}^{b} = (\omega_{[\dot{\mathbf{e}},b]}),{}^{b} = \left(\frac{1}{k}\omega_{\dot{\mathbf{e}},b} - \frac{k-1}{k}\omega_{b[\ddot{\mathbf{e}},e_2]}\right),{}^{b} = \frac{1}{k}\omega_{\dot{\mathbf{e}},b}{}^{b} - \frac{k-1}{k}\omega_{b[\ddot{\mathbf{e}},e_2]}{}^{b}$$

Similarly,

$$\mu_{[\ddot{\mathbf{e}},e_2]} = \left(-\frac{1}{n-k+2}\omega_{b[\ddot{\mathbf{e}},}{}^{b}\right)_{,e_2]} = -\frac{1}{n-k+2}\omega_{b[\ddot{\mathbf{e}},}{}^{b}{}_{e_2]}.$$

Substituting this in the expression for $\tau_{\dot{\mathbf{e}}}$ gives

$$\tau_{\dot{\mathbf{e}}} = -\frac{1}{n}\left[\frac{1}{k}\omega_{\dot{\mathbf{e}},b}{}^{b} - \frac{k-1}{k}\omega_{b[\ddot{\mathbf{e}},e_2]}{}^{b} - \frac{k-1}{n-k+2}\omega_{b[\ddot{\mathbf{e}},}{}^{b}{}_{e_2]} - 2(k-1)P^{b}{}_{[e_2}\omega_{|b|\ddot{\mathbf{e}}]} + P^{b}{}_{b}\omega_{\dot{\mathbf{e}}}\right].$$

We have thus determined the normal splitting operator $L_0 : \Gamma(\Lambda^{k-1}T^*M[k]) \to \Gamma(\Lambda^k\mathcal{T}^*)$ for the kth alternating representation:

$$\begin{aligned} L_0(\omega)_{\mathbf{A}} = \omega_{\dot{\mathbf{b}}}Q_{\mathbf{A}}{}^{\dot{\mathbf{b}}} + \omega_{[\dot{\mathbf{b}},b_1]}R_{\mathbf{A}}{}^{\dot{\mathbf{b}}} - \frac{1}{n-k+2}\omega_{c\ddot{\mathbf{b}},}{}^{c}S_{\mathbf{A}}{}^{\ddot{\mathbf{b}}} \\ - \frac{1}{n}\left[\frac{1}{k}\omega_{\dot{\mathbf{b}},c}{}^{c} - \frac{k-1}{k}\omega_{c[\ddot{\mathbf{b}},b_2]}{}^{c} - \frac{k-1}{n-k+2}\omega_{c[\ddot{\mathbf{b}},}{}^{c}{}_{b_2]} - 2(k-1)P^{c}{}_{[b_2}\omega_{|c|\ddot{\mathbf{b}}]} + P^{c}{}_{c}\omega_{\dot{\mathbf{b}}}\right]T_{\mathbf{A}}{}^{\dot{\mathbf{b}}}. \end{aligned}$$

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