A HEXAGON TILING PROBLEM OF DAN FINKEL

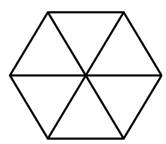
TRAVIS WILLSE

Problem. *How many ways are there to tile a regular hexagon with* 30–60–90 *triangles, each of which has area* $\frac{1}{12}$ *the area of the hexagon?*

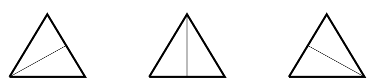
Our strategy is to partition the set of all tilings into natural classes that are relatively easy to count separately and then add the counts. Each class is closed under rotations, allowing us to count the numbers tilings up to rotations for each class separately; this process amounts to analyzing the possible symmetries of tilings in each class, and the counting for each class essentially manually carries out the computation in Burnside's Lemma for counting the number of orbits of a group action. All classes are likewise closed under reflections, except for two which are exchanged under any reflection, again making counting of tilings up to rotation and reflection tractable. To keep the presentation elementary, we henceforth avoid explicit reference to group theory outside footnotes.

1. Tilings

A regular hexagon can be tiled by 6 congruent equilateral arranged in a hexagram.



Many tilings of the hexagon are refinements of this one, and it's straightforward to count these. Each of the 6 triangles can be tiled with two 30–60–90 triangles in exactly three ways—



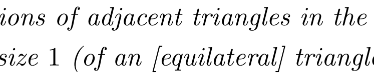
—so there are $3^6 = 729$ such tilings.

Many (as it turns out, most) tilings of the hexagon are not of this form. We count as follows: Each tiling can be built of tilings of unions of adjacent equilateral triangles in the above hexagram. For subsequent use we identify such tilings that are irreducible in the sense that they cannot be built out of such tilings of fewer adjacent equilateral triangles—these tilings are thus in a sense the possible "prime factors" of a tiling. We then identify all possible decompositions of tilings of a hexagon into irreducible tilings.

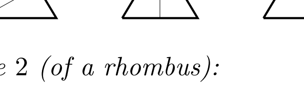
We say that the **size** of an irreducible tiling of n adjacent equilateral triangles in the hexagram is n.

Lemma 1. *The irreducible tilings of unions of adjacent triangles in the above hexagram are as follows.*

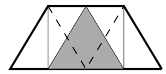
(1) *There are* 3 *irreducible tilings of size* 1 *(of an [equilateral] triangle):*



(2) *There are* 2 *irreducible tilings of size* 2 *(of a rhombus):*

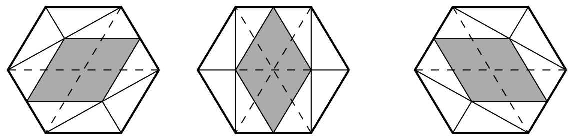


(3) *There are* 3 *irreducible tilings of size* 3 *(of a trapezoid), namely*



where the shaded triangle is one of the 3 *tilings in (1).*

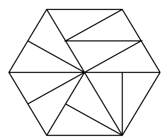
(4) *There are* 6 *irreducible tilings of size* 6 *(of a hexagon), namely,*



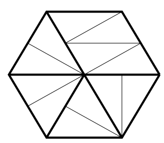
where the shaded rhombus is one of the 2 *tilings in (2).*

We refer to the decomposition of the hexagon into these shapes determined by a tiling T the **pattern** $P(T)$ of the tiling. Denote the set of all tilings of the hexagon by $\mathcal{T}$ and the set of patterns by $\mathcal{P}$. Formally we can regard a pattern as the set $P \subset \mathcal{T}$ of all tilings with that pattern.

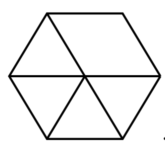
Example 1.1. Decomposing the tiling



into irreducible tilings gives

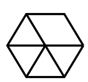


so the tiling has pattern



.

For each pattern we can determine the number of tilings of the hexagon with that pattern by multiplying for each shape the number of irreducible tilings of the corresponding size.

Example 1.2. For the pattern  there are 2 tilings of the rhombus and 3 for each of the four triangles. So there are $2 \cdot 3^4 = 162$ tilings of the hexagon with this pattern.

Rotating any tiling by any number of sixth-turns produces another (not necessarily different) tiling, rotating any pattern produces another (not necessarily different) pattern, and the pattern of a tiling rotated by some amount is the same as the rotation of the pattern of that tiling by the same amount. We define the **class** $K(P) \subset \mathcal{T}$ of a pattern P to be the union of all patterns given by rotations of P and the **class** of a tiling to be the class $K(T) := K(P(T))$ of its pattern.[1]

Our intermediate goal is to identify all possible classes of patterns. To do so, we identify the set of classes with another combinatorial object that makes enumeration more intuitive: Any choice of pattern and choice of initial shape in that pattern determines a partition of 6 into parts of size $1, 2, 3, 6$, and vice versa, by starting at the chosen shape, proceeding anticlockwise, and writing down the size of each shape in the pattern. For a given pattern, changing the choice of initial shape permutes the partition cyclically. Thus, we can identify classes with the equivalence classes of ordered partitions of 6 into parts of size $1, 2, 3, 6$

[1]Rotation of tilings comprises an action of the cyclic group C_6 on $\mathcal{T}$, and that action descends to an C_6-action on $\mathcal{P}$. By definition the class $K(P)$ of a pattern P is $K(P) = \bigcup_{g \in C_6} g \cdot P$, and the class $K(T)$ of a tiling T is $K(T) = \bigcup_{g \in C_6} g \cdot P(T) \subset T$.

modulo cyclic permutation. For uniqueness, we identify a given class by the lexicographically latest cyclic permutation, for convenience we write a partition $a_1 + \cdots + a_n$ as $a_1 \cdots a_n$, and for compactness we denote a string $a \cdots a$ of r a's by a^r but suppress the exponent if $r = 1$.

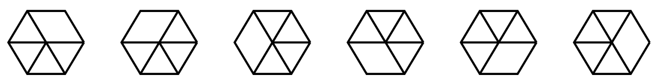
Example 1.3. Taking the pattern  and starting with the rhombus gives the partition $2+1+1+1+1$ of 6, and this partition is evidently the lexicographically latest among its cyclic permutations. So, in our notation the class of this pattern is 21^4.

The classes are:

$$1^6, 21^4, 2121, 2^21^2, 2^3, 31^3, 312, 321, 3^2, 6.$$

For each class the total number of tilings of a hexagon in a given class is the product of the number of tilings with any given pattern in that class and number of patterns in that class. We are now prepared to count the total number of tilings.

Example 1.4. In the class 21^4 there are 6 patterns, namely



and we saw in Example 1.2 that there are 162 tilings with one pattern in that class, so there are $6 \cdot 162 = 972$ tilings in the class.

Repeating the above process for all 10 classes yields the counts in Table 1, and adding gives that the total number of tilings is $\boxed{\mathbf{2\,776}}$.

2. Tilings up to rotation

We now compute the number of tilings of a hexagon up to rotation, that is, where we regard two tilings to be the same if they are related by a rotation.

Our strategy is as follows: First, observe that (by construction) any rotation of a tiling preserves its class, so that we can consider each class separately. Then for each class, we count the number of tilings invariant under rotations of various sizes.[2] The 6 rotations of a typical tiling are distinct in the sense of Section 1, so we expect that the number of tilings up to rotation is approximately $\frac{|\mathcal{T}|}{6} = \frac{2\,776}{6} \approx 463$, but this estimate is a lower bound, since for some tilings the 6 rotations of that tiling are not distinct in the sense of Section 1.

First, note that if the 6 rotations of a tiling $T \in \mathcal{T}$ are not all distinct, neither are the 6 rotations of its pattern, $P(T) \in \mathcal{P}$. By construction the rotations of a pattern exhaust the patterns in its class, which immediately implies that the 6 rotations of all tilings of classes with 6 distinct patterns (21^4, 2^21^2, 31^3, 312, 321) are distinct. Thus, the number of tilings up to rotations of these classes K are respectively $\frac{|K|}{6}$.

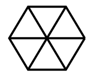
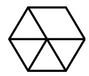
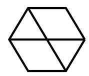
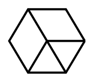
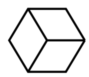
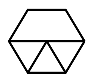
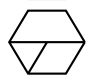
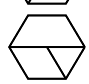
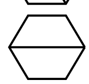
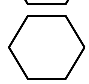
In the remaining classes, not all tilings yield 6 distinct tilings under rotations.

- Class 1^6. This case is the most involved. Since the unique pattern in this class is fixed under rotations, for any given tiling T with this pattern there are $1, 2, 3$, or 6 tilings among its rotations.[3]
  - If all 6 rotations of a tiling preserve a tiling, it is determined precisely by a choice of irreducible tiling of size 1, so there are 3 such patterns.
  - If a tiling is invariant under third-turns, it is determined by 2 adjacent irreducible tilings of size 1. There are $3^2 = 9$ such patterns, and $9-3 = 6$ of them are not also invariant under sixth-turns.
  - If a tiling is invariant under half-turns, it is determined by 3 adjacent irreducible tilings of size 1. There are $3^3 = 27$ such patterns, and $27 - 3 = 24$ of them are not also invariant under sixth-turns.
  - This leaves $729 - 24 - 6 - 3 = 696$ patterns of class 1^6 with 6 distinct patterns among their rotations. The example pattern of this class in Table 2 at the end of the note is of this type.

[2]We can frame our approach naturally using the language of group actions: The rotations of tilings comprise an action of the cyclic group C_6 on the set $\mathcal{T}$ of tilings. Since the action preserves each class K, it restricts to actions C_6 on K, i.e., each class K is a union of C_6-orbits. We then compute the number $|K/C_6|$ of orbits, essentially using Burnside's Lemma, and then compute $|\mathcal{T}/C_6| = \sum_{K \in \mathcal{K}} |K/C_6|$, where $\mathcal{K}$ is the set of classes.

[3]The number of patterns in the class is the size $|C_6 \cdot T| = \frac{|C_6|}{|\mathrm{Stab}_{C_6}(T)|} \in \{1, 2, 3, 6\}$ of the C_6-orbit of T.

TABLE 1. Classes and tiling counts.

class	representative pattern	patterns in class	tilings per pattern	count of tilings
1^6		1	3^6	729
21^4		6	$2 \cdot 3^4$	972
2121		3	$2{\cdot}3{\cdot}2{\cdot}3$	108
$2^2 1^2$		6	$3^2 \cdot 2^2$	216
2^3		2	2^3	16
31^3		6	$3 \cdot 3^3$	486
312		6	$3 \cdot 3 \cdot 2$	108
321		6	$3 \cdot 2 \cdot 3$	108
3^2		3	3^2	27
6		1	6	6
				2 776

So, there are

$$\frac{3}{1} + \frac{6}{2} + \frac{24}{3} + \frac{696}{6} = 3 + 3 + 8 + 116 = 130$$

tilings of class 1^6 up to rotation.[4]

- CLASS 2121. There are 3 patterns in the class, so for any given tiling T of this class the number of distinct patterns among the rotations of T is either 3 or 6. It is 3 iff it is invariant under a half-turn, in which case it is determined precisely by its pattern (3 choices) and (any) 2 adjacent irreducible tilings, which have size $1, 2$. There are thus $3 \cdot 3 \cdot 2 = 18$ tilings of class 2121 invariant under a

[4]We can also compute the count $|(3^1)/C_6|$ of tilings of class 1^6 up to rotation as a straightforward application of Burnside's Lemma: If a finite group G acts on a set X, the count $|X/G|$ of orbits is the average number of fixed points $|X^g|$ over all elements $g \in G$:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

The identity element $e \in G$ fixes all $|(3^1)| = 3^6 = 729$ tilings. Denote (either) sixth-turn by $r \in G$. If a tiling is fixed by sixth-turn r (or r^5) it is determined precisely by the tiling of any 1 triangle, so $|(3^1)^r| = |(3^1)^{r^5}| = 3$. If a tiling is fixed by a third-turn r^2 (or r^4), it is determined precisely by the tiling of any 2 adjacent triangles, so $|(3^1)^{r^2}| = |(3^1)^{r^4}| = 3^2$. If a tiling is fixed by a half-turn r^3, it is determined precisely by the tiling of 3 adjacent triangles, so $|(3^1)^{r^3}| = 3^3$. Thus, the number of tilings of class 1^6 up to rotation is

$$|(3^1)/C_6| = \frac{1}{6}\left(|(3^1)^e| + |(3^1)^r| + \cdots + |(3^1)^{r^5}|\right) = \frac{1}{6}(3^6 + 3 + 3^2 + 3^3 + 3^2 + 3) = 130,$$

in agreement with our manual computation. This computation is a special case of the *necklace formula*.

half-turn and hence $108 - 18 = 90$ tilings of class 2121 that are not. There are thus

$$\frac{18}{3} + \frac{90}{6} = 6 + 15 = 21$$

tilings of class 2121 up to rotation.

- Class 2^3. There are 2 patterns in this class, so for any pattern in this class there are 2 or 6 distinct tilings among its rotations. Any tiling with 2 distinct tilings among its rotations, that is, invariant under a third-turn, is determined by its pattern (2 choices) and a choice of irreducible tiling of size 2, so there are $2 \cdot 2 = 4$ tilings invariant under a third-turn and $16 - 4 = 12$ that are not. There are thus

$$\frac{4}{2} + \frac{12}{6} = 2 + 2 = 4$$

tilings of class 2^3 up to rotation.

- Class 3^2. As for class 2121 there are 3 patterns in the class, so again for any pattern p in this class there are 3 or 6 distinct tilings among its rotations. Any tiling invariant under a half-turn is determined by its pattern (3 choices) and a choice of irreducible tiling of size 3, so there are $3 \cdot 3 = 9$ tilings invariant under a half-turn and $27 - 9 = 18$ that are not, giving

$$\frac{9}{3} + \frac{18}{6} = 3 + 3 = 6$$

tilings of class 3^2 up to rotation.

- Class 6. All 6 patterns in this class are invariant under half-turns and have 3 patterns among their rotations, so there are

$$\frac{6}{3} = 2$$

tilings of class 6 up to rotation.

Adding the counts for each class gives that there are $\boxed{\mathbf{478}}$ tilings up to rotation.

3. Tilings up to rotation and reflection

In principle this case is analogous to the case of tilings up to rotation (but not reflection), and we replace consideration of the 6 rotations by that of all of the rigid symmetries of the regular hexagon, i.e., the 6 rotations together with the 6 reflections. For each class—other than 312 and 321, which are handled efficiently using a straightforward observation—we count the number of tilings up to rotation that are their own mirror image. We call such tilings **achiral** and tilings that are not their own mirror image **chiral**. Then, the number of tilings in the class up to rotation and reflection is the sum of the number of achiral tilings up to rotation and half the number of chiral tilings up to rotation.[5]

The typical tiling is distinct from its mirror image (up to reflection), so we expect that the number of tilings up to rotation is approximately $\frac{478}{2} = 239$, but this estimate is a lower bound, since some tilings are not distinct from their mirror image (up to rotation).

Reflections preserve all classes except that they (all) exchange classes 312 and 321, so we may handle all classes except for those individually.

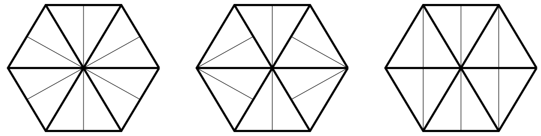
- Classes 312 and 321. No element of either of these classes is fixed (up to a rotation) by a reflection, so the

$$18$$

distinct tilings (up to rotation) in either class exhaust all of the tilings in both classes up to rotation and reflection.

- Class 1^6. Again this class is the most involved. Suppose that a tiling of this class is achiral, so that it has an axis of symmetry. There are two types of axis of symmetry.
  - Firstly, suppose we have an achiral tiling with an axis of symmetry that passes through two opposite vertices of the hexagon. Then, any tiling is determined completely the choice of tilings of the 3 triangles on 1 side of the axis. There are 3^3 of these, but by construction a choice of tilings and its mirror image across the axis perpendicular to the axis of symmetry. There are 3 tilings that are their own mirror images across that axis—corresponding to the tilings

[5]We replace the C_6-action on the regular hexagon with the natural action of D_{12}. As before, this action induces D_{12}-actions on $\mathcal{T}$ and $\mathcal{P}$, and we in particular are interested in the number $|\mathcal{T}/D_{12}|$ of orbits of the action on $\mathcal{T}$.



so there are $\frac{1}{2}(3^3 - 3) + 3 = 15$ achiral tilings of this type.

– Secondly, suppose we have an achiral tiling with an axis of symmetry that passes through the midpoints of two opposite sides of the hexagon. The tiling of the triangles intersecting the axis of symmetry are fixed, and so any tiling of this type is determined by the choice of tilings of the two triangles strictly on one side of the axis of symmetry, for which there are thus 3^2 choices. Again, mirror images of the choices across the axis perpendicular to the ais of symmetry determine the same tilings of the hexagon. By construction the achiral tilings of this type that are their own mirror images are the 3 tilings identified above, so there are $\frac{1}{2}(3^2 - 3) + 3 = 6$ tilings of this type, but exactly the 3 tilings identified above have both types of symmetry.

Accounting for the 3 tilings with both types of symmetry gives that there are

$$15 + 6 - 3 = 18$$

achiral tilings of type 1^6 up to rotation and hence $130 - 18 = 112$ chiral tilings of that type up to rotation. Thus, there are

$$18 + \frac{112}{2} = 18 + 56 = 74$$

tilings of type 1^6 up to rotation and reflection.[6]

- Class 21^4. Any reflection replaces each irreducible tiling of size 2 with the other, so every tiling of this class is chiral, and so there are

$$\frac{162}{2} = 81$$

tilings of class 21^4 up to rotation and reflection.

- Class 2121. Any reflection replaces each irreducible tiling of size 2 with the other, so any achiral tiling contains one of each, and inspecting any pattern in the class shows that the reflection must fix both triangles and hence determines both of them. This leaves just one possibility up to rotation. so there is 1 achiral tiling up to rotation and $21 - 1 = 20$ chiral tilings, giving a total of

$$1 + \frac{20}{2} = 11$$

tilings of class 2121 up to rotation and reflection.

- Class $2^2 1^2$. Any reflection that fixes a tiling necessarily exchanges the triangles and exchanges the rhombi and so an achiral tiling of this class is determined up to rotation precisely by a choice of irreducible tiling of size 1 (3 choices) and a one of size 2 (2 choices). Thus in this class there are $3 \cdot 2 = 6$ achiral tilings and $36 - 6 = 30$ chiral tilings and hence

$$6 + \frac{30}{2} = 6 + 15 = 21$$

[6] Again appealing to Burnside's Lemma handles the count of tilings of class 1^6 efficiently. Let s denote a reflection of the hexagon across a line through 2 opposite vertices. Any tiling fixed by s is determined precisely by the tilings of the 3 triangles on one side of the axis of reflection, hence there are 3^3 such tilings, and the same applies to the reflections $r^2 s, r^4 s$ through the other lines through opposite vertices, so $|(1^6)^s| = |(1^6)^{r^2 s}| = |(1^6)^{r^4 s}| = 3^3$. The remaining reflections, $rs, r^3 s, r^5 s$, are across lines through midpoints of opposite sides. Any tiling fixed by such a reflection is determined precisely by the tilings of the 2 triangles strictly on one side of the axis of reflection, hence for each such reflection there are 3^2 such tilings. So, Burnside's Lemma gives that there are

$$\begin{aligned}
|(1^6)/D_{12}| &= \frac{1}{|D_{12}|} \sum_{g \in D_{12}} |(1^6)^g| \\
&= \frac{1}{12}\left(|(1^6)^e| + |(1^6)^r| + \cdots + |(1^6)^{r^5}| + |(1^6)^s| + |(1^6)^{rs}| + \cdots + |(1^6)^{r^5 s}|\right) \\
&= \frac{1}{12}(3^6 + 2 \cdot 3 + 2 \cdot 3^2 + 3^3 + 3 \cdot 3^3 + 3 \cdot 3^2) \\
&= 74
\end{aligned}$$

tilings of class 1^6 up to rotation and reflection, in agreement with our manual computation.

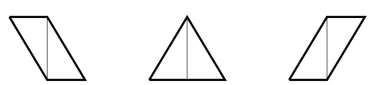
tilings of class $2^2 1^2$ up to rotation and reflection.

- Class 2^3. Any reflection replaces each rhombus tiling with the other and so changes the number of each rhombus tiling. So, every tiling in this class is chiral, leaving just
$$\frac{4}{2} = 2$$
tilings of class 2^3 up to rotation and reflection.
- Class 31^3. Any reflection that fixes a tiling fixes the trapezoid and the central triangle and exchanges the two other triangles, so the trapezoid and the central triangle of an achiral tiling of this class must be symmetric about the line of symmetry, determining both of their tilings up to rotation. Thus, any tiling of this class fixed by a reflection is determined precisely by the tiling of one of the non-central triangles, hence there are 3 achiral tilings (up to rotation) and $81 - 3 = 78$ chiral tilings, giving a total of
$$3 + \frac{78}{3} = 3 + 39 = 42$$
tilings of class 31^3 up to rotation and reflection.
- Class 3^2. A reflection that preserves a pattern of this class either fixes both trapezoids or exchanges them. Two of the tilings are achiral, and the other 4 are achiral, so there
$$2 + \frac{4}{2} = 2 + 2 = 4$$
tilings of class 3^2 up to rotation and reflection.
- Class 6. The 2 tilings in this class up to rotation are mirror images of one another, hence there is just
$$1$$
tiling of class 6 up to rotation and reflection.

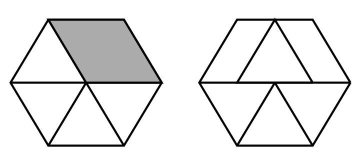
Adding the counts for each class gives that there are $\boxed{\mathbf{254}}$ tilings up to rotation and reflection. A corollary of the above computation is that there are 30 achiral tilings up to rotation (or rotation and reflection).

4. Remark: An alternative approach to counting tilings

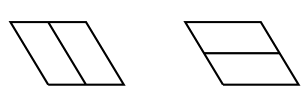
An alternative line of observation quickly leads to the classification of tilings into classes. First, since the sides of the (right) tiling triangle have ratio $1 : \sqrt{3} : 2$, the irrationality of $\sqrt{3}$ implies that in any tiling any long leg must be adjacent to another long leg. Thus, any tiling can be decomposed into an arrangement of 6 atoms of 3 types, namely,



Call them π_1, τ, π_2, respectively, and refer to π_1 and π_2 collectively as π atoms. Any tiling with a π atom has at least 1 short edge along half of an edge of a hexagon. Some meditation shows that, consequently, any tiling has an even number of π atoms and hence an even number $t \in \{0, 2, 4, 6\}$ of τ atoms. Tilings with $t = 6$, i.e., with only triangular atoms, are precisely those of class 1^6. Similarly, up to rotation, some deduction gives that all tilings with $t = 4$ have one of the forms

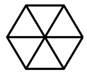
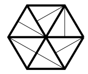
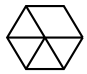
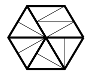
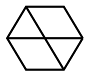
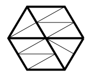
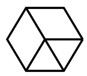
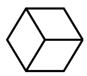
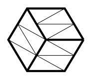
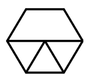
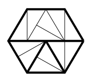
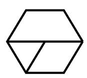
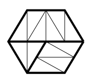
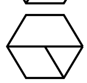
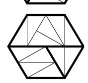
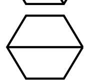
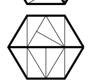
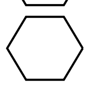
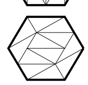


—where the shaded rhombus is one of



(2 π_1 atoms) (2 π_2 atoms)

Table 2. Tiling counts by class

class	representative pattern	typical tiling	patterns in class	tilings per pattern	tiling counts: all	tiling counts: up to rotation	tiling counts: up to rotation /reflection
1^6			1	3^6	729	130	74
21^4			6	$2 \cdot 3^4$	972	162	81
2121			3	$2{\cdot}3{\cdot}2{\cdot}3$	108	21	11
$2^2 1^2$			6	$3^2 \cdot 2^2$	216	36	21
2^3			2	2^3	16	4	2
31^3			6	$3 \cdot 3^3$	486	81	42
312			6	$3 \cdot 3 \cdot 2$	108	18	18 (312 and 321 together)
321			6	$3 \cdot 2 \cdot 3$	108	18	
3^2			3	3^2	27	6	4
6			1	6	6	2	1
					2 776	**478**	**254**

—and these two cases comprise classes 21^4 and 31^3, respectively. In the first case there are 2 choices for the tiling of the rhombus and 3 for the tiling of each of the 4 triangles, and all 6 tilings produced by rotating each choice are distinct, so there are $2 \cdot 3^4 \cdot 6 = 972$ tilings of class 21^4, in agreement with our previous computation. Similar reasoning gives that there are $6 \cdot 3^4 = 486$ tilings of class 31^3.

The tilings with $t = 0$ correspond are those of the classes 2^3 and 6^1, and the tilings of the remaining 5 classes (2121, $2^2 1^2$, 312, 321, 3^2) all have $t = 2$, and we can count the patterns of these classes analogously.