# LEGENDREAN CONTACT GEOMETRY: PARABOLIC GEOMETRY OF TYPE $(A_{n+1}, P_{1,n+1})$

TRAVIS WILLSE

Do not distribute

## 1. Some geometry

Let $(M; \mathbf{E}, \mathbf{F})$ be a Legendrean contact geometry of dimension $2n+1$.

Denote $\mathbf{H} := \mathbf{E} \oplus \mathbf{F}$ and $\mathbf{Q} := TM/\mathbf{H}$, and let $q : TM \to \mathbf{Q}$ denote the canonical projection. Let $\mathcal{L} : \mathbf{E} \times \mathbf{F} \to \mathbf{Q}$ denote the Levi bracket component associated to $\mathfrak{g}^{\mathbf{E}}_{-1} \times \mathfrak{g}^{\mathbf{F}}_{-1} \to \mathfrak{g}_{-2}$, namely, $\mathcal{L}(\xi_p, \eta_p) = q([\xi, \eta]_p)$. Likewise, the Levi bracket component $\mathfrak{g}^{\mathbf{E}}_{-1} \times \mathfrak{g}_{+2} \to \mathfrak{g}^{\mathbf{F}}_{+1}$ induces an isomorphism $\mathbf{E} \otimes \mathbf{Q}^* \cong \mathbf{F}^*$, and tensoring with $\mathbf{Q}$ gives $\mathbf{E} \cong \mathbf{F}^* \otimes \mathbf{Q} \cong \mathbf{F}^*(1,1)$.

## 2. Some key $P_{1,n+1}$-representations

We have

| $\mathbb{W}$ | | bigrading | $n \geq 3$ | $n = 2$ | $n = 1$ |
|---|---|---|---|---|---|
| $\mathfrak{g}_{-2}$ | $\mathbf{Q}$ | $(-1,-1)$ | $\overset{1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!-\!\overset{1}{\times}$ | $\overset{1}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\times}$ | $\overset{1}{\times}\!-\!\overset{1}{\times}$ |
| $\mathfrak{g}^{\mathbf{E}}_{-1}$ | $\mathbf{E}$ | $(-1,0)$ | $\overset{1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{1}{\circ}\!-\!\overset{-1}{\times}$ | $\overset{1}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{-1}{\times}$ | $\overset{2}{\times}\!-\!\overset{-1}{\times}$ |
| $\mathfrak{g}^{\mathbf{F}}_{-1}$ | $\mathbf{F}$ | $(-1,0)$ | $\overset{-1}{\times}\!-\!\overset{1}{\circ}\cdots\overset{0}{\circ}\!-\!\overset{1}{\times}$ | $\overset{-1}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{1}{\times}$ | $\overset{-2}{\times}\!-\!\overset{2}{\times}$ |
| $\mathfrak{g}^{ss}_0$ | $\operatorname{End}_\circ(\mathbf{E}) \cong \operatorname{End}_\circ(\mathbf{F})$ | $(0,0)$ | $\overset{-1}{\times}\!-\!\overset{1}{\circ}\cdots\overset{1}{\circ}\!-\!\overset{-1}{\times}$ | $\overset{-1}{\times}\!-\!\overset{2}{\circ}\!-\!\overset{-1}{\times}$ | - |
| $\mathfrak{g}^{\mathbf{E}}_{+1}$ | $\mathbf{E}^*$ | $(+1,0)$ | $\overset{-2}{\times}\!-\!\overset{1}{\circ}\cdots\overset{0}{\circ}\!-\!\overset{0}{\times}$ | $\overset{-2}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\times}$ | $\overset{-2}{\times}\!-\!\overset{1}{\times}$ |
| $\mathfrak{g}^{\mathbf{F}}_{+1}$ | $\mathbf{F}^*$ | $(0,+1)$ | $\overset{0}{\times}\!-\!\overset{0}{\circ}\cdots\overset{1}{\circ}\!-\!\overset{-2}{\times}$ | $\overset{0}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{-2}{\times}$ | $\overset{1}{\times}\!-\!\overset{-2}{\times}$ |
| $\mathfrak{g}_{+2}$ | $\mathbf{Q}^*$ | $(+1,+1)$ | $\overset{-1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!-\!\overset{-1}{\times}$ | $\overset{-1}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{-1}{\times}$ | $\overset{-1}{\times}\!-\!\overset{-1}{\times}$ |
| $\bigwedge^n \mathfrak{g}^{\mathbf{E}}_{-1}$ | $\bigwedge^n \mathbf{E}$ | $(-n,0)$ | $\overset{n+1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!-\!\overset{-1}{\times}$ | $\overset{3}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{-1}{\times}$ | $\overset{2}{\times}\!-\!\overset{-1}{\times}$ |
| $\bigwedge^n \mathfrak{g}^{\mathbf{F}}_{-1}$ | $\bigwedge^n \mathbf{F}$ | $(0,-n)$ | $\overset{-1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!-\!\overset{n+1}{\times}$ | $\overset{-1}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{3}{\times}$ | $\overset{-1}{\times}\!-\!\overset{2}{\times}$ |
| $\bigwedge^n \mathfrak{g}^{\mathbf{E}}_{+1}$ | $\bigwedge^n \mathbf{E}^*$ | $(+n,0)$ | $\overset{-n-1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!-\!\overset{1}{\times}$ | $\overset{-3}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\times}$ | $\overset{-2}{\times}\!-\!\overset{1}{\times}$ |
| $\bigwedge^n \mathfrak{g}^{\mathbf{F}}_{+1}$ | $\bigwedge^n \mathbf{F}^*$ | $(0,+n)$ | $\overset{1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!-\!\overset{-n-1}{\times}$ | $\overset{1}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{-3}{\times}$ | $\overset{1}{\times}\!-\!\overset{-2}{\times}$ |

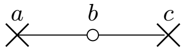
We denote $\mathcal{E}(a,b) := \overset{a}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!-\!\overset{b}{\times}$. Then,

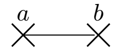
$$\mathcal{E}(+1,+1) = \mathbf{Q}$$
$$\mathcal{E}(-1,-1) = \mathbf{Q}^*$$
$$\mathcal{E}((n+2)n, 0) = (\wedge^n \mathbf{E})^{\otimes(n+1)} \otimes (\wedge^n \mathbf{F})$$
$$\mathcal{E}(0, (n+2)n) = (\wedge^n \mathbf{E}) \otimes (\wedge^n \mathbf{F})^{\otimes(n+1)}.$$

The bigrading element is

$$\overset{a_1}{\times}\!-\!\overset{a_2}{\circ}\cdots\overset{a_n}{\circ}\!-\!\overset{a_{n+1}}{\times} \quad \mapsto$$
$$(-\tfrac{1}{n+2}[(n+1)a_1 + na_2 + \cdots + 2a_n + a_{n+1}], -\tfrac{1}{n+2}[a_1 + 2a_2 + + \cdots + na_n + (n+1)a_{n+1}])$$

for $n \geq 3$, and for $n = 1, 2$ it is



$\mapsto (-\frac{1}{4}(3a + 2b + c), -\frac{1}{4}(a + 2b + 3c))$,


$\mapsto (-\frac{1}{3}(2a + b), -\frac{1}{3}(a + 2b))$.

## 3. Index notation

We use the index notation from Cap & Slovak, § 5.2.15: We use lowercase Greek letters, $\alpha, \beta, \gamma, \ldots$, for indices on $\mathbf{E}$, and write $\xi^\alpha \in \Gamma(\mathbf{E})$ and $\eta_\alpha \in \Gamma(\mathbf{E}^*)$. We use barred lowercase Greek letters, $\bar\alpha, \bar\beta, \bar\gamma, \ldots$, for indices on $\mathbf{F}$, and write $\xi^{\bar\alpha} \in \Gamma(\mathbf{F})$ and $\eta_{\bar\alpha} \in \Gamma(\mathbf{F}^*)$.

We use lozenges, $\lozenge, \blacklozenge$, for indices on $\mathbf{Q}$, and write $\xi^\lozenge \in \Gamma(\mathbf{Q})$ and $\xi_\lozenge \in \Gamma(\mathbf{Q}^*)$. When it does not result in ambiguity, we sometimes suppress a lozenge index.

Using the isomorphism $\mathbf{E} \cong \mathbf{F}^*(1, 1)$ and its dual map we can change the position and barredness of an index at the cost of a symmetric change of biweight; we denote these isomorphisms $\mathcal{L}_{\alpha\bar\beta}{}^\lozenge$ (or $\mathcal{L}_{\alpha\bar\beta}$) and $\mathcal{L}^{\alpha\bar\beta}{}_\lozenge$ (or $\mathcal{L}^{\alpha\bar\beta}$). In particular, we can realize the isomorphism $\operatorname{End}(\mathbf{E}) \cong \operatorname{End}(\mathbf{F})$ via $\phi^\alpha{}_\beta \mapsto \phi_{\bar\alpha}{}^{\bar\beta}$.

## 4. Integrable second-order systems of PDEs

Integrable Legendrean contact geometries are locally realizable as (integrable) systems of second-order PDEs, that is, integrable PDE systems of the form

$$u_{x^i x^j} = f_{ij}(x^k, (x^k)')$$

(here, $f_{ji} = f_{ij}$). On the appropriate jet space we use the variables $p_k$ for the derivatives $(x^k)'$. If we denote the total derivative with respect to $x^i$ by

$$D_{x^i} := \partial_{x^i} + f_{ij}\partial_{p_j} + p_i \partial_u,$$

then we set

$$\mathbf{E} := \operatorname{span}\{D_{x^1}, \ldots, D_{x^n}\}, \qquad \mathbf{F} := \operatorname{span}\{\partial_{p_1}, \ldots, \partial_{p_n}\}.$$

These distributions are invariant under point transformations. Integrability of the PDE system is equivalent to integrability of $\mathbf{E}$.

Lie brackets of elements of the adapted frame $(\partial_u, D_{x^i}, \partial_{p_i})$ are given by

$$\begin{aligned}
[\partial_u, D_{x^i}] &= (\partial_u f_{ij})\partial_{p_j} \\
[\partial_u, \partial_{p_i}] &= 0 \\
[D_{x^i}, D_{x^j}] &= 0 \\
[D_{x^i}, \partial_{p_j}] &= -(\partial_{p_j} f_{ik})\partial_{p_k} - \delta^j{}_i \partial_u \\
[\partial_{p_i}, \partial_{p_j}] &= 0.
\end{aligned}$$

The coframe dual to $(\partial_u, D_{x^i}, \partial_{p_i})$ is $(\sigma, \theta^i, \pi_i)$, where

$$\begin{aligned}
\sigma &= du - p_i\, dx^i \\
\theta^i &= dx^i \\
\pi_i &= dp_i - f_{ij} dx^j.
\end{aligned}$$

Adaptation implies that $\sigma$ is a contact form for $\mathbf{H}$; differentiating gives

$$d\sigma = dx^i \wedge dp_i.$$

The only nontrivial evaluations of $d\sigma$ on basis elements are those determined by

$$d\sigma(D_{x^i}, \partial_{p_i}) = 1.$$

We denote the volume $n$-vectors

$$\operatorname{vol}_{\mathbf{E}} := D_{x^1} \wedge \cdots \wedge D_{x^n} \in \Gamma(\textstyle\bigwedge^n \mathbf{E}), \qquad \operatorname{vol}_{\mathbf{F}} := \partial_{p_1} \wedge \cdots \wedge \partial_{p_n} \in \Gamma(\textstyle\bigwedge^n \mathbf{F}).$$

## 5. The partial connections $\nabla^{\mathbf{E}}, \nabla^{\mathbf{F}}$

We use $\mathbf{Q}^*$ as a bundle of scales; pulling back a nonvanishing section of $\mathbf{Q}^*$ via $TM \to \mathbf{Q}$ gives a contact form $\sigma$ of $\mathbf{H}$. Equation (5.17) in C & S gives that the restrictions $\nabla^{\mathbf{E}}, \nabla^{\mathbf{F}}$ of the Weyl connection associated to $\sigma$ are characterized by

$$\begin{aligned}
d\sigma(\nabla^{\mathbf{E}}_{\eta_1}\xi, \eta_2) &= d\sigma([\eta_1, \xi], \eta_2) \\
d\sigma(\nabla^{\mathbf{E}}_{\xi_1}\xi_2, \eta) &= \xi_1 \cdot d\sigma(\xi_2, \eta) + d\sigma(\xi_2, [\xi_1, \eta]) \\
d\sigma(\nabla^{\mathbf{F}}_{\xi_1}\eta, \xi_2) &= d\sigma([\xi_1, \eta], \xi_2) \\
d\sigma(\nabla^{\mathbf{F}}_{\eta_1}\eta_2, \xi) &= \eta_1 \cdot d\sigma(\eta_2, \xi) + d\sigma(\eta_2, [\eta_1, \xi]).
\end{aligned}$$

5.1. **Integrable systems of PDEs.** Taking $\sigma|_{\mathbf{Q}} \in \Gamma(\mathbf{Q}^*)$ as our scale gives the contact form $\sigma$ for $\mathbf{H}$.

Using earlier formulae gives (herein abusing notation slightly by denoting also by $\theta^j$ and $\pi_j$ respectively the pullbacks of $\theta^j$ and $\pi_j$ via $\mathbf{H} \hookrightarrow TM$.)

$$\begin{aligned}
\nabla^{\mathbf{E}} D_{x^i} &= (-\partial_{p_k} f_{ij}) D_{x^k} \otimes \theta^j \\
\nabla^{\mathbf{F}} \partial_{p_i} &= (-\partial_{p_i} f_{jk}) \partial_{p_k} \otimes \theta^j.
\end{aligned}$$

Dualizing gives formulae for the dual connections:

$$\begin{aligned}
\nabla^{\mathbf{E}} \theta^i &= (\partial_{p_i} f_{jk}) \theta^j \otimes \theta^k \\
\nabla^{\mathbf{F}} \pi_i &= (\partial_{p_j} f_{ik}) \pi_j \otimes \theta^k.
\end{aligned}$$

The induced connections on the bundles of $n$-vectors and $n$-forms are then characterized by

$$\begin{aligned}
\nabla^{\mathbf{E}} \operatorname{vol}_{\mathbf{E}} &= -(\partial_{p_i} f_{ij}) \operatorname{vol}_{\mathbf{E}} \otimes \theta^j \\
\nabla^{\mathbf{F}} \operatorname{vol}_{\mathbf{F}} &= -(\partial_{p_i} f_{ij}) \operatorname{vol}_{\mathbf{F}} \otimes \theta^j.
\end{aligned}$$

The induced connection on the line bundle $\mathcal{E}(a, b)$ is

$$\nabla \epsilon = -\left(\frac{a+b}{n}\right) (\partial_{p_i} f_{ij}) \epsilon \otimes \theta^j,$$

where $\epsilon \in \Gamma(\mathcal{E}(a, b))$ satisfies $\epsilon^{(n+2)n} := (\wedge^n \mathbf{E})^{\otimes(an+a+b)} \otimes (\wedge^n \mathbf{F})^{\otimes(bn+a+b)}$.

## 6. BGG operators, $n > 1$

6.1. **Dual second exterior power of the standard representation: $\wedge^2 \mathbb{V}^*$.** By Kostant, the operator is

$$\Theta : \Gamma\left(\overset{0}{\times}\!\!-\!\!\!-\!\!\overset{1}{\circ}\cdots\overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\times}\right) \to \Gamma\left(\overset{-2}{\times}\!\!-\!\!\!-\!\!\overset{2}{\circ}\cdots\overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\times} \oplus \overset{0}{\times}\!\!-\!\!\!-\!\!\overset{1}{\circ}\cdots\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{-2}{\times}\right),$$

or

$$\Theta : \Gamma(\mathbf{E}^*(2, 0)) \to \Gamma(S^2\mathbf{E}^*(2, 0) \oplus (\mathbf{E}^* \otimes \mathbf{F}^*)_\circ(2, 0)).$$

(For $n = 2$, it is

$$\Theta : \Gamma\left(\overset{0}{\times}\!\!-\!\!\!-\!\!\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\times}\right) \to \Gamma\left(\overset{-2}{\times}\!\!-\!\!\!-\!\!\overset{2}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\times} \oplus \overset{0}{\times}\!\!-\!\!\!-\!\!\overset{2}{\circ}\!\!-\!\!\!-\!\!\overset{-2}{\times}\right).)$$

The respective component maps $\Theta^{\mathbf{E}}, \Theta^{\mathbf{F}}$ are both first-order operators, so they are given by differentiating by the restriction of the Weyl connection to the appropriate subbundle and projecting onto the highest-weight irreducible subbundle:

$$\begin{aligned}
\Theta^{\mathbf{E}} &: \mu_\alpha \mapsto \nabla_{(\alpha}\mu_{\beta)} \\
\Theta^{\mathbf{F}} &: \mu_\alpha \mapsto (\nabla_{\bar{\alpha}}\mu_\beta)_\circ = \nabla_{\bar{\alpha}}\mu_\beta - \frac{1}{n}(\mathcal{L}^{\gamma\bar{\zeta}}\nabla_{\bar{\zeta}}\mu_\gamma)\mathcal{L}_{\beta\bar{\alpha}}.
\end{aligned}$$

6.1.1. *Integrable systems of PDEs.* For $\mu = \mu_j \theta^j \otimes \epsilon$:

- The components of $\Theta^{\mathbf{E}}(\mu)_{\alpha\beta}$ are
$$\frac{1}{2}(D_{x^i} \cdot \mu_j + D_{x^j} \cdot \mu_i) + \frac{n-2}{2} \alpha_k \partial_{p_k} f_{ij}.$$
(NB the second term vanishes for $n = 2$.)
- The components of $\Theta^{\mathbf{F}}(\mu)_{\alpha\bar{\beta}}$ are
$$\partial_{p_i} \cdot \mu_j - \frac{1}{n}(\partial_{p_k} \cdot \mu_k) \delta^i{}_j,$$
that is, the tracefree part of $\partial_{p_i} \cdot \mu_j$.

6.2. **Second exterior power of the standard representation: $\wedge^2 \mathbb{V}$.** By Kostant, the operator is

$$\Theta : \Gamma\left( \overset{0}{\times}\!\!-\!\!\overset{0}{\circ}\cdots\overset{1}{\circ}\!\!-\!\!\overset{0}{\times} \right) \to \Gamma\left( \overset{0}{\times}\!\!-\!\!\overset{0}{\circ}\cdots\overset{2}{\circ}\!\!-\!\!\overset{-2}{\times} \oplus \overset{-2}{\times}\!\!-\!\!\overset{1}{\circ}\cdots\overset{1}{\circ}\!\!-\!\!\overset{0}{\times} \right),$$

or

$$\Theta : \Gamma(\mathbf{F}^*(0,2)) \to \Gamma(S^2\mathbf{F}^*(0,2) \oplus (\mathbf{E}^* \otimes \mathbf{F}^*)_\circ(0,2)).$$

For $n = 2$, $\wedge^2\mathbb{V} \cong \wedge^2\mathbb{V}^*$; see the relevant subsection.

The respective component maps $\Theta^{\mathbf{E}}, \Theta^{\mathbf{F}}$ are both first-order operators, so they are given by differentiating by the restriction of the Weyl connection to the appropriate subbundle and projecting onto the highest-weight irreducible subbundle:

$$\Theta^{\mathbf{E}} : \mu_{\bar{\alpha}} \mapsto (\nabla_\alpha \mu_{\bar{\beta}})_\circ = \nabla_\alpha \mu_{\bar{\beta}} - \frac{1}{n}(\mathcal{L}^{\gamma\bar{\zeta}} \nabla_\gamma \mu_{\bar{\zeta}}) \mathcal{L}_{\alpha\bar{\beta}}$$
$$\Theta^{\mathbf{F}} : \mu_{\bar{\alpha}} \mapsto \nabla_{(\bar{\alpha}} \mu_{\bar{\beta})}.$$

6.2.1. *Integrable systems of PDEs.* For $\mu = \mu^j \pi_j \otimes \epsilon$:

- The components of $\Theta^{\mathbf{E}}(\mu)_{\bar{\alpha}\bar{\beta}}$ are
$$\frac{1}{2}(\partial_{p_i} \cdot \mu^j + \partial_{p_j} \cdot \mu^i).$$
- The components of $\Theta^{\mathbf{F}}(\mu)_{\alpha\bar{\beta}}$ are
$$D_{x^i} \cdot \mu^j + \mu^k \partial_{p_j} f_{ik} - \frac{2}{n} \mu^j \partial_{p_k} f_{ik} - \frac{1}{n}\left(D_{x^k} \cdot \mu^k + \frac{n-2}{n} \mu^k \partial_{p_l} f_{kl}\right) \delta^j{}_i,$$
that is, the tracefree part of $D_{x^i} \cdot \mu^j + \mu^k \partial_{p_j} f_{ik} - \frac{2}{n}\mu^j \partial_{p_k} f_{ik}$. Again the formula simplifies for $n = 2$.

6.3. **Rho tensor.** In homogeneity 2, the harmonic curvature takes values in $\wedge^2_\circ \mathfrak{g}_{-1} \otimes \mathfrak{g}_0^{ss}$.

Precomputing gives:

(1) $$[[(X,Y),(Z,W)],r] = -[B(X,Z) + B(Y,W)]r$$

(2) $$[[(X,Y),(Z,W)],(X',Y')] = [B(X',Z)X - B(X',Y)W + B(X,Z)X', B(Y,W)Y' - B(X,Y')Z + B(W,Y')Y].$$

Here, $B$ is $\frac{1}{2(n+2)}$ times the Killing form.

The adjoint action of $\mathfrak{g}_0^{ss}$ on $\mathfrak{g}_{+2}$ is trivial, so $\{(R^{(2)} + \partial P^{(2)})(\xi,\eta), q(r) = 0$, and since $\nabla(q(r)) = 0$, we get $0 = \{\{\xi, \mathsf{P}^{(2)}(\eta)\} - \{\eta, \mathsf{P}^{(2)}(\xi)\}, q(r)\}$, so substituting in (1) and rearranging gives $\mathsf{P}^{(2)}(\eta)(\xi) = \mathsf{P}^{(2)}(\xi)(\eta)$; decomposition according to the splitting $\mathbf{H} = \mathbf{E} \oplus \mathbf{F}$ gives precisely the identities

(3) $$\mathsf{P}_{\alpha\beta} = \mathsf{P}_{\beta\alpha}, \qquad \mathsf{P}_{\alpha\bar{\beta}} = \mathsf{P}_{\bar{\beta}\alpha}, \qquad \mathsf{P}_{\bar{\alpha}\bar{\beta}} = \mathsf{P}_{\bar{\beta}\bar{\alpha}}.$$

The decomposition $\mathbf{H} = \mathbf{E} \oplus \mathbf{F}$ determines decompositions of $\wedge^2\mathbf{H}^* \otimes \operatorname{End}_\circ(\mathbf{H})$. In particular, (2) display equation shows that the components of $\{\,\cdot\,, \mathsf{P}(\,\cdot\,)\}$ taking values in $L(\mathbf{E},\mathbf{F})$ and $L(\mathbf{F},\mathbf{E})$ are zero. Specializing the above formula gives that the remaining components of $\{\,\cdot\,, \mathsf{P}(\,\cdot\,)\} : \mathbf{H}^* \otimes \mathbf{H}^* \to \operatorname{End}_\circ(\mathbf{H})$ are

$$\begin{aligned}
\mathbf{E} \times \mathbf{E} \to \operatorname{End}(\mathbf{E}) : &\qquad \mathsf{P}_{\beta\zeta}\delta^\gamma{}_\alpha + \mathsf{P}_{\beta\alpha}\delta^\gamma{}_\zeta \\
\mathbf{E} \times \mathbf{E} \to \operatorname{End}(\mathbf{F}) : &\qquad -\mathcal{L}_{\alpha\bar\zeta}\mathsf{P}_\beta{}^{\bar\gamma} \\
\mathbf{E} \times \mathbf{F} \to \operatorname{End}(\mathbf{E}) : &\qquad \mathsf{P}_{\bar\beta\zeta}\delta^\gamma{}_\alpha + \mathsf{P}_{\bar\beta\alpha}\delta^\gamma{}_\zeta \\
\mathbf{E} \times \mathbf{F} \to \operatorname{End}(\mathbf{F}) : &\qquad -\mathcal{L}_{\alpha\bar\zeta}\mathsf{P}_{\bar\beta}{}^{\bar\gamma} \\
\mathbf{F} \times \mathbf{E} \to \operatorname{End}(\mathbf{E}) : &\qquad -\mathcal{L}_{\zeta\bar\alpha}\mathsf{P}_\beta{}^\gamma \\
\mathbf{F} \times \mathbf{E} \to \operatorname{End}(\mathbf{F}) : &\qquad \mathsf{P}_{\beta\bar\alpha}\delta^{\bar\gamma}{}_{\bar\zeta} + \mathsf{P}_{\beta\bar\zeta}\delta^{\bar\gamma}{}_{\bar\alpha} \\
\mathbf{F} \times \mathbf{F} \to \operatorname{End}(\mathbf{E}) : &\qquad -\mathcal{L}_{\zeta\bar\alpha}\mathsf{P}_{\bar\beta}{}^\gamma \\
\mathbf{F} \times \mathbf{F} \to \operatorname{End}(\mathbf{F}) : &\qquad \mathsf{P}_{\bar\beta\bar\alpha}\delta^{\bar\gamma}{}_{\bar\zeta} + \mathsf{P}_{\bar\beta\bar\zeta}\delta^{\bar\gamma}{}_{\bar\alpha}.
\end{aligned}$$

Thus, the part of $\partial\mathsf{P}$ with components in $\bigwedge^2\mathbf{H} \to \operatorname{End}_\circ(\mathbf{H})$ (in particular using (3)) has $\operatorname{End}(\mathbf{E})$-valued components

$$\begin{aligned}
\textstyle\bigwedge^2\mathbf{E}^* \otimes \operatorname{End}(\mathbf{E}) : &\qquad (\partial\mathsf{P})_{\alpha\beta}{}^\gamma{}_\zeta = \mathsf{P}_{\beta\zeta}\delta^\gamma{}_\alpha - \mathsf{P}_{\alpha\zeta}\delta^\gamma{}_\beta \\
\mathbf{E}^* \otimes \mathbf{F}^* \otimes \operatorname{End}(\mathbf{E}) : &\qquad (\partial\mathsf{P})_{\alpha\bar\beta}{}^\gamma{}_\zeta = \mathsf{P}_{\bar\beta\alpha}\delta^\gamma{}_\zeta + \mathsf{P}_{\bar\beta\zeta}\delta^\gamma{}_\alpha + \mathcal{L}_{\zeta\bar\beta}\mathsf{P}_\alpha{}^\gamma \\
\textstyle\bigwedge^2\mathbf{F}^* \otimes \operatorname{End}(\mathbf{E}) : &\qquad (\partial\mathsf{P})_{\bar\alpha\bar\beta}{}^\gamma{}_\zeta = -\mathcal{L}_{\zeta\bar\alpha}\mathsf{P}_{\bar\beta}{}^\gamma + \mathcal{L}_{\zeta\bar\beta}\mathsf{P}_{\bar\alpha}{}^\gamma,
\end{aligned}$$

and the $\operatorname{End}(\mathbf{F})$-valued components can be recovered from the isomorphism $\operatorname{End}(\mathbf{E}) \cong \operatorname{End}(\mathbf{F})$. (*** Is this: $\phi^\alpha{}_\beta \mapsto \phi_{\bar\alpha}{}^{\bar\beta} = -\phi^{\bar\beta}{}_{\bar\alpha}$?):

$$\begin{aligned}
\textstyle\bigwedge^2\mathbf{E}^* \otimes \operatorname{End}(\mathbf{F}) : &\qquad (\partial\mathsf{P})_{\alpha\beta}{}^{\bar\gamma}{}_{\bar\zeta} = -\mathcal{L}_{\alpha\bar\zeta}\mathsf{P}_\beta{}^{\bar\gamma} + \mathcal{L}_{\beta\bar\zeta}\mathsf{P}_\alpha{}^{\bar\gamma} \\
\mathbf{E}^* \otimes \mathbf{F}^* \otimes \operatorname{End}(\mathbf{F}) : &\qquad (\partial\mathsf{P})_{\alpha\bar\beta}{}^{\bar\gamma}{}_{\bar\zeta} = -\mathsf{P}_{\alpha\bar\beta}\delta^{\bar\gamma}{}_{\bar\zeta} - \mathsf{P}_{\alpha\bar\zeta}\delta^{\bar\gamma}{}_{\bar\beta} - \mathcal{L}_{\alpha\bar\zeta}\mathsf{P}_{\bar\beta}{}^{\bar\gamma} \\
\textstyle\bigwedge^2\mathbf{F}^* \otimes \operatorname{End}(\mathbf{F}) : &\qquad (\partial\mathsf{P})_{\bar\alpha\bar\beta}{}^{\bar\gamma}{}_{\bar\zeta} = \mathsf{P}_{\bar\beta\bar\zeta}\delta^{\bar\gamma}{}_{\bar\alpha} - \mathsf{P}_{\bar\alpha\bar\zeta}\delta^{\bar\gamma}{}_{\bar\beta}
\end{aligned}$$

Computing gives that the image of $A \in \Gamma(\bigwedge^2\mathbf{H}^* \otimes \operatorname{End}(\mathbf{E}))$ under the codifferential $\partial^* : \bigwedge^2\mathbf{H}^* \otimes \operatorname{End}(\mathbf{H}) \to \mathbf{H}^* \otimes \operatorname{End}(\mathbf{H})$ has components

$$\begin{aligned}
(\partial^* A)_{\alpha\beta} &= A_{\gamma\alpha}{}^\gamma{}_\beta \\
(\partial^* A)_{\alpha\bar\beta} &= A_{\alpha\bar\gamma\bar\beta}{}^{\bar\gamma} \\
(\partial^* A)_{\bar\alpha\beta} &= A_{\gamma\bar\alpha}{}^\gamma{}_\beta \\
(\partial^* A)_{\bar\alpha\bar\beta} &= A_{\bar\alpha\bar\gamma\bar\beta}{}^{\bar\gamma}.
\end{aligned}$$

Now, expanding the normality condition $\partial^*(R^{(2)} + \partial\mathsf{P}^{(2)}) = 0$ gives

$$\begin{aligned}
\mathsf{P}_{\alpha\beta} &= -\frac{1}{n-1}R_{\gamma\alpha}{}^\gamma{}_\beta \\
\mathsf{P}_{\alpha\bar\beta} &= -\frac{1}{n+1}\left(-R_{\bar\gamma\alpha}{}^{\bar\gamma}{}_{\bar\beta} + \frac{1}{2n+1}R_{\bar\gamma\zeta}{}^{\bar\gamma\zeta}\mathcal{L}_{\alpha\bar\beta}\right) \\
\mathsf{P}_{\bar\alpha\bar\beta} &= -\frac{1}{n-1}R_{\bar\gamma\bar\alpha}{}^{\bar\gamma}{}_{\bar\beta}
\end{aligned}$$

6.4. **Transformation rules.** We can write any change of scale as $\sigma \rightsquigarrow \widehat{\sigma} = \sigma(\exp\Upsilon_1)(\exp\Upsilon_2)$.

*Email address*: `travis.willse@univie.ac.at`