n-GONAL NUMBERS THAT ARE SQUARES OF n-GONAL NUMBERS

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The nth s-gonal number, for integers $s \geq 3, n \geq 1$, is

$$P(s,n) := \frac{1}{2}[(s-2)n^2 - (s-4)n].$$

We wish to find all s-gonal numbers that are squares of other s-gonal numbers, that is, all triples (s,m,n) such that

$$P(s,n) = P(s,m)^2,$$

which defines a quartic curve in m,n. Since $P(4,n) = n^2$, the solutions for $s = 4$ are precisely $n = m^2$, $m \geq 1$. For convenience we use the parameter $t := s - 2$.

Denoting $U = m$ and substituting

$$V = 2nt - (t-2)$$

and rearranging gives

$$V^2 = \underbrace{2t^3U^4 - 4(t-2)t^2U^3 + 2(t-2)^2tU^2 + (t-2)^2}_{Q_{t+2}(U)}$$

The discriminant of $Q_{t+2}(U)$ is $\operatorname{disc}_s = [2^4(t-2)^3t^4(t+2)]^2$, so for $t \neq -2, 0, 2$, $V^2 = Q_{t+2}(U)$ defines an elliptic curve $E_s = E_{t+2}$. The transformation

$$U = \frac{2(t-2)[3x + 4(t-2)^2t]}{3d(x)},$$
$$V = \frac{(t-2)[54(x^3 + 2(t-2)^2tx^2) - 27(y^2 + 8(t-2)^2t^2y) - 16(t-2)^4t^3(4t^2 + 11t + 16)]}{27d(x,y)^2},$$

where $d(x,y) := y + 4(t-2)^2t^2$, places it in Weierstrass short form,

$$y^2 = \underbrace{x^3 - \frac{4}{3}(t-2)^2t^2(t^2 + 2t + 4)x + \frac{16}{27}(t-2)^4t^3(t+1)(s+2)}_{q_{t+2}(x)},$$

or as

$$y^2 = \left(x - \left(-\frac{4}{3}(t-2)t(t+1)\right)\right)\left(x - \frac{2}{3}(t-2)^2t\right)\left(x - \frac{2}{3}(t-2)t(t+4)\right).$$

The j-invariant of this curve is $j_s = 64(t^2 + 2t + 4)^3/t^2(t+2)^2$, and its discriminant is $\Delta := 2^4 \operatorname{disc}_t = [2^6(t-2)^3t^4(t+2)]^2$.

For $t \geq 3$, we denote the roots of $q_{t+2}(x)$ by

$$e_1 = \frac{2}{3}(t-2)t(t+2), \qquad e_2 = \frac{2}{3}(t-2)^2t, \qquad e_3 = -\frac{4}{3}(t-2)t(t+1),$$

and for $t = 1$ we denote them by

$$e_1 = \frac{8}{3}, \qquad e_2 = \frac{2}{3}, \qquad e_3 = -\frac{10}{3}.$$

In all cases, $e_1 > e_2 > e_3$, the elements $t_i := (e_i, 0)$, $i = 1,2,3$ are torsion elements of order 2, and $\operatorname{Tor}(E_s(\mathbb{Q})) \geq \{\mathcal{O}, t_1, t_2, t_3\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that if $s \not\equiv 0 \pmod 3$ then q_1, q_2, q_3 are integral with respect to short form $y^2 = q(x)$.

Remark 1 (Galois group of Q_s)**.** For $s = 3$ and $s > 5$, $\operatorname{Gal}(Q_s) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and $\operatorname{Gal}(Q_s)$ acts transitively: For any numbering of the roots, $\operatorname{Gal}(Q_s) = \langle \mathrm{id}, (12)(34), (13)(24), (14)(23) \rangle$.

The first claim is equivalent to the irreducibility of Q_t: If $(t+2)^2$ is not a power of 2, then $Q_{t+2}(U-1)$ is Eisenstein at any odd prime factor of $t+2$. If $(t+2)^2$ is a power of 2, then for $t \neq 6$, $(t-2)^2$ is not a power of 2 and $Q_{t+2}(U)$ is Eisenstein at any odd prime factor of $t-2$. Finally, $Q_8(U-1) = 16(27U^4 + 72U^3 + 66U^2 + 24U + 4)$, and the quantity in parentheses is Eisenstein at 2.

Since Δ_t is a square, $\operatorname{Gal}(Q_{t+2}) \leq A_4$. Finally, for any t, the resolvent cubic of Q_{t+2} is

$$R_{t+2}(z) = t^2z(t^2z + 2t - 4)(tz - t + 2)$$

In particular, R_{t+2} is reducible.

TABLE 1. Elliptic curves E_s for admissible $s \leq 2^4$

s	minimal model	isomorphism	rank	discriminant	conductor	Cremona	free generators	torsion generators
3	$Y^2 = X^3 - X^2 - 9X + 9$	$(1, \frac{1}{3})$	1	$2^{12} \cdot 3^2$	$2^6 \cdot 3$	192a2	$(5, 8)$	$(3, 0), (1, 0)$
5	$Y^2 = X^3 - 228X + 448$	$(1, 0)$	1	$2^{12} \cdot 3^8 \cdot 5^2$	$2^6 \cdot 3^2 \cdot 5$	2880z2	$(29, 135)$	$(14, 0), (2, 0)$
6	$Y^2 = X^3 - X^2 - 9X + 9$	$(\frac{1}{4}, \frac{1}{3})$	1	$2^{12} \cdot 3^2$	$2^6 \cdot 3$	192a2	$(5, 8)$	$(3, 0), (1, 0)$
7	$Y^2 = X^3 - 11\,300X + 32\,400$	$(1, 0)$	1	$2^{12} \cdot 3^6 \cdot 5^8 \cdot 7^2$	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7$	100800fo2	$(130, 1\,000)$	$(90, 0), (30, 0)$
8	$Y^2 = X^3 - 156X + 560$	$(\frac{1}{4}, 0)$	1	$2^{14} \cdot 3^8$	$2^6 \cdot 3^2$	576i2	$(13, 27)$	$(10, 0), (4, 0)$
9	$Y^2 = X^3 - X^2 - 109\,433X + 11\,215\,737$	$(1, \frac{1}{3})$	1	$2^{12} \cdot 3^4 \cdot 5^6 \cdot 7^8$	$2^6 \cdot 3 \cdot 5^2 \cdot 7^2$	235200gr2	$(313, 2\,744)$	$(257, 0), (117, 0)$
10	$Y^2 = X^3 - 63X + 162$	$(\frac{1}{8}, 0)$	1	$2^8 \cdot 3^6 \cdot 5^2$	$2^3 \cdot 3^2 \cdot 5$	360e2	$(9, 18)$	$(6, 0), (3, 0)$
11	$Y^2 = X^3 - X^2 - 6\,729X + 187\,209$	$(\frac{1}{3}, \frac{1}{3})$	2	$2^{12} \cdot 3^4 \cdot 7^6 \cdot 11^2$	$2^6 \cdot 3 \cdot 7^2 \cdot 11$	103488bf2	$(69, 216), (83, 440)$	$(61, 0), (33, 0)$
12	$Y^2 = X^3 - X^2 - 258X + 1\,512$	$(\frac{1}{8}, \frac{1}{3})$	1	$2^6 \cdot 3^2 \cdot 5^8$	$2^5 \cdot 3 \cdot 5^2$	2400a1	$(132, 1\,500)$	$(12, 0), (7, 0)$
13	$Y^2 = X^3 - 23\,716X + 1\,277\,760$	$(\frac{1}{3}, 0)$	1	$2^{12} \cdot 11^8 \cdot 13^2$	$2^6 \cdot 11^2 \cdot 13$	100672ck2	$(1\,397, 51\,909)$	$(110, 0), (66, 0)$
14	$Y^2 = X^3 - 12\,900X + 520\,000$	$(\frac{1}{4}, 0)$	2	$2^{12} \cdot 3^8 \cdot 5^6 \cdot 7^2$	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7$	100800oa2	$(86, 216), (150, 1400)$	$(80, 0), (50, 0)$
15	$Y^2 = X^3 - X^2 - 5\,425\,801X + 4\,538\,445\,001$	$(1, \frac{1}{3})$	2	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 11^6 \cdot 13^8$	$2^6 \cdot 3 \cdot 5 \cdot 11^2 \cdot 13^2$	-	$(1\,725, 17\,576), (2\,831, 108\,900)$	$(1\,621, 0), (1\,049, 0)$
16	$Y^2 = X^3 - 33\,516X + 2\,222\,640$	$(\frac{1}{4}, 0)$	1	$2^{16} \cdot 3^6 \cdot 7^8$	$2^6 \cdot 3^2 \cdot 7^2$	28224ck2	$(133, 343)$	$(126, 0), (84, 0)$

(The smallest s for which $\operatorname{rank} E_s = 3$ is $s = 37$, and the smallest for which $\operatorname{rank} E_s = 4$ is $s = 101$.)
A pair (u, r) specifies an isomorphism $(X, Y) \mapsto (x, y) = (u^2 X + r, u^3 Y)$ from the minimal model of E_s to E_s itself.

TABLE 2. Elliptic curves E_s for select inadmissible s

s	minimal model	isomorphism	rank	discriminant	conductor	Cremona	free generators	torsion generators
-2	$Y^2 = X^3 - 36X$	$(\frac{1}{4}, 0)$	1	$2^{12} \cdot 3^6$	$2^6 \cdot 3^2$	576h2	$(12, 36)$	$(6, 0), (0, 0)$
-1	$Y^2 = X^3 - 2\,100X + 20\,000$	$(1, 0)$	2	$2^{12} \cdot 3^8 \cdot 5^6$	$2^6 \cdot 3^2 \cdot 5^2$	14400bt2	$(46, 144), (50, 200)$	$(40, 0), (10, 0)$
1	$Y^2 = X^3 - 36X$	$(1, 0)$	1	$2^{12} \cdot 3^6$	$2^6 \cdot 3^2$	576h2	$(12, 36)$	$(6, 0), (0, 0)$

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