Almost Einstein $(2, 3, 5)$ conformal structures

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Basic definitions

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Definition

A (pseudo-)Riemannian metric g is Einstein if $\mathrm{Ric} = \mu g$.

Problem: When does a conformal structure $\mathbf{c}$ contain an Einstein metric g? (Brinkmann, Hanntjes & Wrona, et al.)

The conformal-to-Einstein problem

- We follow Bailey, Eastwood, Gover: Given $g \in \mathbf{c}$, we search for a $\hat{g}$ with $\widehat{\mathrm{Ric}} = \mu \hat{g}$, that is, $\widehat{\mathrm{Ric}}_\circ = 0$ (here, $\cdot_\circ$ denotes tracefree part).

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- This pde is nonlinear in Ω but...

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Theorem (Bailey-Eastwood-Gover ('94))

For any conformal structure **c**, *there is a bijective correspondence*

$$\{\textit{almost Einstein scales } \sigma\} \underset{\Pi_0^{\mathcal{V}}}{\overset{L_0^{\mathcal{V}}}{\rightleftarrows}} \{\nabla^{\mathcal{V}}\textit{-parallel sections } \mathbb{S} \textit{ of } \mathcal{V}\}$$

(2, 3, 5) distributions

Definition

A $(2,3,5)$ distribution $\mathbf{D}$ is a 2-plane field on a 5-manifold M satisfying the genericity condition that $[\mathbf{D},[\mathbf{D},\mathbf{D}]] = TM$ (this implies that $\operatorname{rank}[\mathbf{D},\mathbf{D}] = 3$).

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A $(2,3,5)$ distribution $(M,\mathbf{D})$ canonically determines a conformal structure $\mathbf{c}_{\mathbf{D}}$ (of signature $(2,3)$) on M.

- We call the conformal structures that arise this way $(2,3,5)$ **conformal structures**

(2, 3, 5) conformal structures (cont.)

- Question: When does a $(2, 3, 5)$ conformal structure $(M, \mathbf{c_D})$ admit an almost Einstein scale?

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An (oriented) conformal structure $\mathbf{c}$ *is induced by a distribution* $\mathbf{D}$ *if* $\mathcal{V}$ *admits a* $\nabla^{\mathcal{V}}$*-parallel tractor* G_2^**-structure compatible with the conformal structure in the sense that* $G_2^* < SO(H) \cong SO(3,4)$.

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- Such a G_2^*-structure can be encoded in a $\nabla^{\mathcal{V}}$ parallel tractor 3-form $\Phi \in \Gamma(\Lambda^3\mathcal{V}^*)$.

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- Such a G_2^*-structure can be encoded in a $\nabla^{\mathcal{V}}$ parallel tractor 3-form $\Phi \in \Gamma(\Lambda^3 \mathcal{V}^*)$.
- Underlying ϕ is a (weighted) 2-form $\phi \in \Gamma(\Lambda^2 T^* M \otimes \mathcal{E}[3])$. This turns out to be locally decomposable and nowhere zero, so it defines a 2-plane field on M, namely $\mathbf{D}$

(Almost Einstein $(2,3,5)$ conformal structures)

Proposition

An oriented conformal structure of signature $(2,3)$ is both $(2,3,5)$ and almost Einstein iff it admits a holonomy reduction to the intersection S of $\mathrm{Stab}(\Phi) = \mathrm{G}_2^*$ and the stabilizer $\mathrm{Stab}(\mathbb{S})$ of a nonzero vector in $\mathbb{V}$.

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- There are three cases according to the causality type of the vector
  - (spacelike) $S = \mathrm{SU}(1,2)$, g Ricci-negative
  - (isotropic) $S = \mathrm{SL}(2,\mathbb{R}) \ltimes Q_+$, g Ricci-flat
  - (timelike) $S = \mathrm{SL}(3,\mathbb{R})$, g Ricci-positive

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- Raising an index of $-\mathbb{S} \lrcorner \Phi$ gives a parallel H-skew endomorphism $\mathbb{K} \in \Gamma(\operatorname{End} \mathcal{V})$. Underlying it is a conformal Killing field $\xi \in \Gamma(TM)$ of $\mathbf{c}$.

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- The flow Ξ_t of ξ does *not* preserve $\mathbf{D}$, hence the images of $\mathbf{D}$ under Ξ_t comprise a 1-parameter family $\{\mathbf{D}_t\}$ whose elements all induce the same conformal structure.

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- This connects the existence of an almost Einstein scale for $\mathbf{c}_{\mathbf{D}}$ with a *conformal isometry problem*:

Problem

Given a $(2, 3, 5)$ *distribution* $\mathbf{D}$, *what are the distributions* $\mathbf{D}'$ *such that* $\mathbf{c}_{\mathbf{D}'} = \mathbf{c}_{\mathbf{D}}$*? Alternatively, what are the fibers of Nurowski's functor,* $\mathbf{D} \rightsquigarrow \mathbf{c}_{\mathbf{D}}$*?*

Compatible 3-forms

- Working at the level of parallel sections of tractor bundles, this problem becomes algebraic: What are the 3-forms Φ' that (1) are stabilized by the common stabilizer S of $\mathbb{S}$ and Φ and (2) are compatible with H ($\mathrm{G}_2^* \cong \mathrm{Stab}_{\mathrm{SO}(H)}(\Phi') < \mathrm{SO}(H)$). May as well assume $\varepsilon := -H(\mathbb{S}, \mathbb{S}) \in \{-1, 0, +1\}$.

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- These are precisely the 3-forms

$$\Phi' := \Phi + \bar{A}\, \mathbb{S} \lrcorner (\mathbb{S}^\flat \wedge \Phi) + B\, \mathbb{S} \lrcorner * \Phi,$$

where $-\varepsilon \bar{A}^2 + 2\bar{A} + B^2 = 0$

Solution to the conformal isometry problem

Theorem

Fix an oriented $(2,3,5)$ *distribution* $(M, \mathbf{D})$.

1. *Suppose* $(M, \mathbf{c_D})$ *admits the nonzero almost Einstein scale* σ*; by rescaling, we may assume that* $\varepsilon \in \{-1, 0, +1\}$*. Then, for* $(\bar{A}, B)$ *as before,*

$$\begin{aligned}\phi'_{ab} := \phi_{ab} + \bar{A}\left[\tfrac{1}{5}\sigma^2\left(\tfrac{1}{3}\phi_{ab,c}{}^{c} + \tfrac{2}{3}\phi_{c[a,b]}{}^{c} + \tfrac{1}{2}\phi_{c[a,}{}^{c}{}_{b]} + 4\mathsf{P}^{c}{}_{[a}\phi_{b]c}\right) - \sigma\sigma^{,c}\phi_{[ca,b]}\right. \\ \left. - \tfrac{1}{2}\sigma\sigma_{,[a}\phi_{b]c,}{}^{c} - \tfrac{1}{5}\sigma\sigma_{,c}{}^{c}\phi_{ab} + 3\sigma^{,c}\sigma_{,[c}\phi_{ab]}\right] \\ + B[-\tfrac{1}{4}\sigma\phi^{cd,}{}_{d}\phi_{[ab,c]} + \tfrac{3}{4}\sigma^{,c}\phi_{[ab}\phi_{c]d,}{}^{d}]\end{aligned}$$

   determines a $\mathbf{D}'$ *for which* $\mathbf{c_{D'}} = \mathbf{c_D}$.

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   determines a $\mathbf{D}'$ for which $\mathbf{c}_{\mathbf{D}'} = \mathbf{c}_{\mathbf{D}}$.

2. *Conversely, all conformally isometric oriented $(2,3,5)$ distributions arise this way.*

Theorem

For any $\mathbf{D}$, the oriented conformal structure $\mathbf{c}_{\mathbf{D}}$ is almost Einstein if there is a $\mathbf{D}' \neq \mathbf{D}$ such that $\mathbf{c}_{\mathbf{D}'} = \mathbf{c}_{\mathbf{D}}$.

Cartan geometries

- Both (oriented) conformal geometry and (oriented) $(2, 3, 5)$ distributions can be realized as *Cartan geometries* of certain types.

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- For a pair (G, P), $P \leq G$ of Lie groups, these are pairs $(\mathcal{G} \to M, \omega)$, where $\mathcal{G}$ is a principal P-bundle and ω is a *Cartan connection*. Modeled on $(G \to G/P, \omega_{\mathsf{MC}})$. Idea: Use these to encode underlying geometric structures on M.

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- Oriented conformal geometry in signature $(2,3)$: Type $(\mathrm{SO}(3,4), \bar{P})$
- Oriented $(2,3,5)$ conformal structures: Type (G_2^*, Q).
- For a conformal structure $\mathbf{c}$ with corresponding Cartan geometry $(\bar{\mathcal{G}}, \bar{\omega})$, the standard tractor bundle is $\bar{\mathcal{G}} \times_{\bar{P}} \mathbb{V}$ and ω induces $\nabla^{\mathcal{V}}$, so a holonomy reduction of $\nabla^{\mathcal{V}}$ determines a holonomy reduction of $(\bar{\mathcal{G}}, \bar{\omega})$.

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- Modeled on S-orbit decomposition of the flat model G/P.
- Each curved orbit M_a inherits a principal bundle embedding $j_a : \mathcal{G}_a \hookrightarrow \mathcal{G}|M_a$, and $(\mathcal{G}_a, j_a^*\omega)$ is a Cartan geometry of type $(H, P \cap S)$—curved orbits correspond to conjugacy classes of $P \cap S$. We can try to interpret these new Cartan geometries in terms of underlying data.

Curved orbits for a.E. $(2, 3, 5)$ conformal structures

Ricci-negative case

M_a	$(S, S \cap P)$	structure	σ	$\mathbf{L} := \langle \xi \rangle$
$M_5^{\pm}$	$(\mathrm{SU}(1,2), \mathrm{SU}(1,1))$	Sasaki-Einstein $(-\sigma^{-2} g, \xi)$	$\pm\sigma > 0$	$\mathbf{L} \subset [\mathbf{D}, \mathbf{D}]$ $\mathbf{L} \pitchfork \mathbf{D}$
M_4	$(\mathrm{SU}(1,2), P_-)$	Fefferman $(1, 3)$ conformal structure	$\sigma = 0$	$\mathbf{L} \subset \mathbf{D}$

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- Conversely, given (L^4, h, J), we can build (Fefferman) an $\mathbb{S}^1$-bundle $M \to L$ canonically equipped with $\mathbf{c}$, σ, $\{\mathbf{D}_t\}$.

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- Conversely, given (L^4, h, J), we can build (Fefferman) an $\mathbb{S}^1$-bundle $M \to L$ canonically equipped with $\mathbf{c}$, σ, $\{\mathbf{D}_t\}$.
- Likewise, $\mathbf{L}|_{M_4}$ is the distinguished symmetry of a Fefferman conformal structure, and its (3d) leaf space inherits a CR-structure.

A Ricci-negative example

Example

Take $(\mathbb{S}_\pm, h_\pm, J_\pm)$ to be the round sphere and hyperbolic plane with their usual Riemann surface structures, normalized so that their scalar curvatures are ± 12. Then, $(S_+ \times S_-, h_+ \oplus -h_-, J_+ \oplus J_-)$ is a Kähler manifold satisfying $\mathrm{Ric} = 6(h_+ \oplus -h_-)$. The 1-parameter family of distributions on the twistor space M^5 are all equivalent under the flow of ξ, and are each diffeomorphic to the rolling distribution $\mathbf{D}$ ('no-slip, no-twist') for S_+ on S_- on the rolling configuration space for S_+ and S_-. $\mathrm{Hol}(\nabla^{\mathcal{V}}) \cong \mathrm{SU}(1,2)$.

Curved orbits for a.E. $(2, 3, 5)$ conformal structures II

Ricci-positive case

M_a	$(S, S \cap P)$	structure	σ	$\mathbf{L} := \langle \xi \rangle$
$M_5^{\pm}$	$(\mathrm{SL}(3,\mathbb{R}), \mathrm{SL}(2,\mathbb{R}))$	para–Sasaki-Einstein $(-\sigma^{-2} g, \xi)$	$\pm\sigma > 0$	$\mathbf{L} \subset [\mathbf{D}, \mathbf{D}]$ $\mathbf{L} \pitchfork \mathbf{D}$
M_4	$(\mathrm{SL}(3,\mathbb{R}), P_{12})$	para-Fefferman $(2, 2)$ conformal structure	$\sigma = 0$	$\mathbf{L} \subset \mathbf{D}$
$M_2^{\pm}$	$(\mathrm{SL}(2,\mathbb{R}), P)$	projective surface	$\sigma = 0$	$\xi = 0$

Curved orbits for a.E. $(2,3,5)$ conformal structures II

Ricci-positive case

M_a	$(S, S \cap P)$	structure	σ	$\mathbf{L} := \langle \xi \rangle$
$M_5^\pm$	$(\mathrm{SL}(3,\mathbb{R}), \mathrm{SL}(2,\mathbb{R}))$	para–Sasaki-Einstein $(-\sigma^{-2} g, \xi)$	$\pm\sigma > 0$	$\mathbf{L} \subset [\mathbf{D}, \mathbf{D}]$ $\mathbf{L} \pitchfork \mathbf{D}$
M_4	$(\mathrm{SL}(3,\mathbb{R}), P_{12})$	para-Fefferman $(2,2)$ conformal structure	$\sigma = 0$	$\mathbf{L} \subset \mathbf{D}$
$M_2^\pm$	$(\mathrm{SL}(2,\mathbb{R}), P)$	projective surface	$\sigma = 0$	$\xi = 0$

- Similar to Ricci-negative case. This time, the leaf space L^3 of M^4 inherits a 3d Lagrangean contact structure, equivalently an o.d.e. $y'' = F(x, y, y')$ modulo point equivalence.

Curved orbits for a.E. $(2,3,5)$ conformal structures II

Ricci-positive case

M_a	$(S, S \cap P)$	structure	σ	$\mathbf{L} := \langle \xi \rangle$
$M_5^{\pm}$	$(\mathrm{SL}(3,\mathbb{R}), \mathrm{SL}(2,\mathbb{R}))$	para–Sasaki-Einstein $(-\sigma^{-2} g, \xi)$	$\pm\sigma > 0$	$\mathbf{L} \subset [\mathbf{D}, \mathbf{D}]$ $\mathbf{L} \pitchfork \mathbf{D}$
M_4	$(\mathrm{SL}(3,\mathbb{R}), P_{12})$	para-Fefferman $(2,2)$ conformal structure	$\sigma = 0$	$\mathbf{L} \subset \mathbf{D}$
$M_2^{\pm}$	$(\mathrm{SL}(2,\mathbb{R}), P)$	projective surface	$\sigma = 0$	$\xi = 0$

- Similar to Ricci-negative case. This time, the leaf space L^3 of M^4 inherits a 3d Lagrangean contact structure, equivalently an o.d.e. $y'' = F(x, y, y')$ modulo point equivalence.
- This isolates an interesting class of a.E. $(2,3,5)$ conformal structures. For any projective surface $(N, \mathbf{p})$, consider the geodesic o.d.e. and build the para-Fefferman structure it determines. Then, try to find a collar.

Thank you.