Sasaki-Einstein metrics and their compactifications via projective geometry

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Sasaki geometry

- A Sasaki structure on a manifold (M, g) is a complex structure $\mathbb{J}$ on its metric cone $(\widehat{M} := M \times \mathbb{R}_+, h := dr^2 + r^2 g)$ such that $(h, \mathbb{J})$ is a Kähler structure on $\widehat{M}$.

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- We can use the homogeneity of the structure to translate this into a condition on M. In particular, identify $M \leftrightarrow M \times \{1\}$ and set $k := \mathbb{J}(r\partial_r)|_M \in \Gamma(TM)$.

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Definition

A Sasaki manifold is a triple (M, g, k) where g is a metric, and $k \in \Gamma(TM)$ satisfies

(0) $g_{ab} k^a k^b = 1$

(1) $\nabla_{(a} k_{b)} = 0$ (equivalently, $\mathcal{L}_k g = 0$, i.e., k Killing)

(2) $\nabla_a \nabla_b k^c = -g_{ab} k^c + \delta^c{}_a k_b$

Sasaki geometry via projective

- The curvature R of a (t.-f., special) affine connection ∇ decomposes as

$$R_{ab}{}^{c}{}_{d} = \underbrace{W_{ab}{}^{c}{}_{d}}_{\text{Weyl}} + 2\delta^{c}{}_{[a}\underbrace{\mathrm{P}_{b]d}}_{\text{Schouten}}$$

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Theorem (Gover-Neusser-W)

A triple (M, g, k) defines a Sasaki manifold iff

(0) $g_{ab}k^{a}k^{b} = 1$

(1) $\nabla_{(a}k_{b)} = 0$ *(k Killing)*

(2) $W_{ab}{}^{c}{}_{d}k^{d} = 0$ *and* $\mathsf{P}_{ab}k^{a}k^{b} = 1$.

Prolonging the Killing-type equation $\nabla_{(a}k_{b)} = 0$

- For any $k_a \in \Gamma(T^*M)$, set $\mu_{ab} := \nabla_{[a}k_{b]} = 0 \in \Gamma(\wedge^2 T^*M)$.

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- So, $\nabla_{(a}k_{b)} = 0$ is equivalent to

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Proposition

Solutions of $\nabla_{(a}k_{b)} = 0$ are in 1-1 correspondence with sections of $T^*M \oplus \wedge^2 T^*M$ parallel w.r.t.

$$\nabla_a^{\mathrm{prol}} \begin{pmatrix} k_b \\ \mu_{bc} \end{pmatrix} := \begin{pmatrix} \nabla_a k_b - \mu_{bc} \\ \nabla_a \mu_{bc} + 2\mathrm{P}_{a[b}k_{c]} \end{pmatrix} - \begin{pmatrix} 0 \\ W_{bc}{}^d{}_a k_d \end{pmatrix}.$$

Projective geometry: A one-slide primer

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A **projective structure** on a manifold M is an equivalence class $\mathbf{p} := [\nabla]$ of t.-f. connections where $\nabla \sim \nabla'$ iff ∇, ∇' have the same unparameterized geodesics.

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- Define the **cotractor bundle** $\mathcal{T}^* := J^1\mathcal{E}(1)$:

$$0 \to T^*M \otimes \mathcal{E}(1) \to \mathcal{T}^* \to \mathcal{E}(1) \to 0$$

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$$0 \to T^*M(1) \to \mathcal{T}^* \to \mathcal{E}(1) \to 0 \qquad B(w) := B \otimes \mathcal{E}(w)$$

- $\exists$ a canonical **normal (co)tractor connection** $\nabla^{\mathcal{T}}$ on $\mathcal{T}^*$

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- Remark: $\mathcal{D}^{\wedge^2\mathcal{T}^*} : k_a \mapsto \nabla_{(a}k_{b)}$ is a **BGG operator**.

The Thomas cone

- There is a canonical line bundle $M_\# \to M$ (the total space is the **Thomas cone**) such that $TM_\#$ factors over the **(standard) tractor bundle** $\mathcal{T} := (\mathcal{T}^*)^*$:

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- Then, since $\nabla^{\mathcal{T}} h = 0$, $\mathrm{Hol}(\nabla^{\mathcal{T}}) \subseteq \mathrm{SO}(h)$.

Einstein-Sasaki geometry via projective holonomy

- If (M, g, k) is Sasaki-Einstein, there is $\mathbb{J} \in \operatorname{End} \mathcal{T}$ compatible with g and satisfying $\nabla^{\mathcal{T}} \mathbb{J} = 0$.

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Theorem

If (M, g, k) is Sasaki-Einstein, the projective holonomy of the projective structure $[\nabla^g]$ carries a ***parallel tractor Hermitian structure*** *$(h, \mathbb{J})$; equivalently, its holonomy satisfies*

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- In fact, if M simply connected, there is a parallel complex tractor volume form, so that $\mathrm{Hol}(\nabla^{\mathcal{T}}) \subseteq \mathrm{SU}(p, q)$ (alternatively, work with restricted holonomy).

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- We can also ask this for each of the intermediate groups:

$$\begin{array}{ccccccc} \mathrm{U}(p,q) & = & \mathrm{SO}(2p,2q) & \cap & \mathrm{Sp}(2m+2,\mathbb{R}) & \cap & \mathrm{GL}(m+1,\mathbb{C}) \\ & & h & & \Omega & & \mathbb{J} \end{array}$$

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(Actually, in the last case, $\mathrm{SL}(2m+2, \mathbb{R}) \cap \mathrm{GL}(m+1, \mathbb{C}) = \mathrm{SL}(m+1, \mathbb{C}) \times \mathrm{U}(1)$.)

Holonomy reductions of projective structures

- $SO(2p, 2q)$, tractor metric h (Cap-Gover-Hammerl)
  - Open orbits $M_{\pm}$: Einstein (pseudo-)Riemannian structure
  - Separating hypersurface M_0: Conformal structure (yields compactification!)
    If h definite ($pq = 0$) then $M = M_{\pm}$.
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  - Underlying torsion-free contact projective structure on M (generalized Fefferman construction).
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- $\mathrm{GL}(m+1, \mathbb{C})$, tractor complex structure $\mathbb{J}$ (Armstrong)
  - Locally fibers (with model fiber U(1)) over an integrable c-projective structure.*
  - Underlying object: Normal solution k^a of adjoint BGG operator.
    (*Under additional condition: k^a is a projective symmetry, equivalently, $W_{da}{}^b{}_c k^d = 0$—always satisfied in Sasaki-Einstein case.)

Projective structures w parallel tractor Hermitian structures

Theorem (Gover-Neusser-W)

Let $(M, \mathbf{p})$ *be a projective manifold of odd dim.* $2m + 1 \geq 5$ *equipped with a parallel tractor Hermitian structure* $(h, \mathbb{J})$*, equivalently, a holonomy reduction of* $\nabla^{\mathcal{T}}$ *to* $\mathrm{U}(p, q)$*.*
Then, M *is stratified as* $M_+ \cup M_0 \cup M_-$ *(if* h *is definite,* $M = M_\pm$*, and* $M_0 = \emptyset$*), and* $\mathbb{J}$ *determines an underlying projective symmetry* k*. The components of the stratifications each inherit a geometry canonically determined by* $(M, \mathbf{p}; h, \mathbb{J})$*.*

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- *$M_\pm$ are open and equipped with Sasaki-Einstein structures $(g_\pm, k)$ with Ricci curvature $\mathrm{Ric}_\pm = 2mg_\pm$; g_+ has signature $(2p-1, 2q)$, and g_- has signature $(2q-1, 2p)$. The metrics $g_\pm$ are compatible with $\mathbf{p}$ in that $\nabla^{g_\pm} \in \mathbf{p}|_{M_\pm}$.*

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- *M_0 is a smooth separating hypersurface and is equipped with an oriented (local) Fefferman conformal structure $\mathbf{c}$ of signature $(2p-1, 2q-1)$. In particular, $(M_0, \pm\mathbf{c})$ is a projective infinity for $(M_\pm, g_\pm)$.*

The (local) fibration by k

- In the previous picture, we can form the local leaf space $\widetilde{M}$ of integral curves of k; it inherits a c-projective structure $(J, \mathbf{q})$. Since k is tangent to M_0, we get a stratification $\widetilde{M} = \widetilde{M}_+ \cup \widetilde{M}_0 \cup \widetilde{M}_-$:

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  - $\widetilde{M}_\pm$ are open and equipped with Kähler-Einstein metrics $\tilde{g}_\pm$.
  - $\widetilde{M}_0$ inherits from a CR-structure; the Fefferman conformal structure it induces is $\mathbf{c}$.

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- From Sasaki-Einstein structure (M, g, k), build the parallel tractor data $h, \mathbb{J}$ on the underlying projective structure $(M, [\nabla^g])$:

$$h := \begin{pmatrix} g_{ab} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{J} := \begin{pmatrix} \nabla_b k^a & k^a \\ -k_b & 0 \end{pmatrix}$$

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Proof (Sketch).

Build Fefferman conformal construction, carry out Fefferman-Graham ambient metric construction, interpret in projective terms, use parallel tractor extension result. □

Thank you.