Special geometries via projective holonomy

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- A.R. Gover, R. Panai, W, Nearly Kähler Geometry and $(2,3,5)$ distributions via projective holonomy, *Indiana Univ. Math. J.* **66**(4) (2017), 1351–1416. arXiv:1403.1959
- A.R. Gover, K. Neusser, W, Projective geometry of Sasaki-Einstein structures and their compactification (submitted). arXiv:1803.09531

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Example ((Oriented) Euclidean space)

The isometry group of $(\mathbb{R}^n, \bar{g})$ is $G := \mathrm{SO}(n, \mathbb{R}) \ltimes \mathbb{R}^n$, which acts transitively with stabilizer $P := \mathrm{SO}(n, \mathbb{R})$, so $G/H \cong \mathbb{R}^n$.

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- Later: We want to generalize (G, P) to 'curved verions' of that geometry (*Cartan geometries*) in the same way that Riemannian metrics (M, g) generalize $(\mathbb{R}^n, \bar{g})$. In this context, we call (G, P) the **flat model** of the geometry it defines.

Projective (differential) geometry

Definition

A **projective structure** on a smooth manifold M is an equivalence class $\mathbf{p} = [\nabla]$ of torsion-free connections on M, where $\nabla \sim \hat{\nabla}$ iff $\nabla, \hat{\nabla}$ share the same geodesics.

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- $\nabla \sim \hat{\nabla}$ iff there is an $\Upsilon \in \Gamma(T^*M)$ such that

$$\hat{\nabla}_a \eta_b = \nabla_a \eta_b + \Upsilon_a \eta_b + \Upsilon_b \eta_a \quad \forall \eta \in \Gamma(T^*M).$$

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Example (Projective sphere)

Let $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{S}^n$ denote the ray projectivization. There is a connection $\bar{\nabla}$ for which the geodesics are precisely (the arcs of) the great circles, that is, the circles $\pi(\Pi - \{0\})$, $\Pi \in G(2, \mathbb{R}^{n+1})$.

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- $G/P := \mathrm{SL}(n+1, \mathbb{R})/P = \{\text{space of rays}\} = \mathbb{S}^n$
- G maps 2d subspaces $\Pi \subset \mathbb{R}^{n+1}$ to 2-planes, so it preserves lines (great circles) in $\mathbb{S}^n$, i.e., the **flat projective structure** $\bar{\mathbf{p}} := [\bar{\nabla}]$.

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- The H-action realizes each orbit X_a is a homogeneous space (H, P_a).
- Morally $P_a = H \cap P$, and more precisely, the H-orbits parameterize the intersections of conjugates of H with P up to conjugacy.

Example: Projective geometry, orthogonal reduction

$$G = \mathrm{SL}(n+1, \mathbb{R}), \quad P = P_{\mathrm{SL}}, \quad H = \mathrm{SO}(p, q).$$

- CASE I: If form is definite ($p = 0$ or $q = 0$), then $H \cong \mathrm{SO}(n, \mathbb{R})$ acts transitively on $\mathbb{S}^n$ with stabilizer $P_a = P_{\mathrm{SL}} \cap H = \mathrm{SO}(n-1, \mathbb{R})$.

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  $$(\mathrm{SO}(n,\mathbb{R}), \mathrm{SO}(n-1,\mathbb{R}))$$

  Bilinear form induces round metric on $\mathrm{SO}(n,\mathbb{R})/\mathrm{SO}(n-1,\mathbb{R}) \cong \mathbb{S}^n$, and that metric is preserved exactly by $H = \mathrm{SO}(n,\mathbb{R})$.

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Remark

The curved version of $(\mathrm{SO}(n,\mathbb{R}), \mathrm{SO}(n-1,\mathbb{R}))$ *is also Riemannian geometry, but with the round metric as its model.*

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- Corresponding geometry: Oriented conformal geometry of signature $(p-1, q-1)$.)

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$$\omega_{MC}(X_g) := T_g L_{g^{-1}} \cdot X_g \in T_e G \cong \mathfrak{g}.$$

- ω_{MC} satisfies nice properties:
  - P-equivariance
  - $(\omega_{MC})_g : T_g G \xrightarrow{\cong} \mathfrak{g}$
  - Maps invariant vector field generated by $X \in \mathfrak{h}$ to X.

Cartan geometry

- A *Cartan geometry* of type (G, P) is a pair $(\mathcal{G} \to M, \omega)$ where $\mathcal{G}$ is a P-principal bundle and ω is a *Cartan connection* thereon, namely, a form $\omega \in \Gamma(T^*G \otimes \mathfrak{g})$ satisfying the conditions on the previous page (replacing $\mathcal{G}$ with G).

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- Main advantages:
  - Treats a wide variety of geometric structures in a common framework.
  - When a canonical construction $(\text{structure}) \rightsquigarrow \omega$ exists, ω encodes higher-order data of the structure and can be used to construct invariants, most importantly, *curvature* $\Omega := d\omega + \frac{1}{2}[\omega, \omega]$ (vanishes for (G, ω_{MC})).

Holonomy of a Cartan geometry

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- Though ω is not a principal connection, it extends uniquely equivariantly to a principal connection $\widehat{\omega}$ on $\widehat{\mathcal{G}} := \mathcal{G} \times_P G$. Then, declare $\mathrm{Hol}(\omega) := \mathrm{Hol}(\widetilde{\omega})$, so $\mathrm{Hol}(\omega)$ is a subgroup (more precisely, a conjugacy class of subgroups) in G.

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- Q: For $(\mathcal{G}, \omega)$ of type (G, P), what are the geometric consequences of fixing a subgroup (conjugacy class thereof) $\mathrm{Hol}(\omega) \leq H \leq G$?

Alternative formulation: tractor geometry

- Fix $(\mathcal{G}, \omega)$. For any representation $\mathbb{W}$ of G, consider the associated vector bundle $\mathcal{W} := \mathcal{G} \times_P \mathbb{W}$; ω induces a vector bundle connection $\nabla^{\mathcal{W}}$, and $\mathsf{Hol}(\nabla^{\mathcal{W}}) \cong \mathsf{Hol}(\omega)$.

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Remark

In the case of projective geometry, the tractor bundle determined by $\mathbb{V}^* = (\mathbb{R}^{n+1})^*$ is canonically isomorphic to $J^1\mathcal{E}(1)$, where $\mathcal{E}(1)$ is the bundle of densities of projective weight 1.

Curved orbit decomposition

- (Not maximally general but covers many important cases:) Suppose for some representation $\mathbb{W}$ that $s \in \Gamma(\mathcal{W})$ is a $\nabla^{\mathcal{W}}$-parallel, so that $\mathrm{Hol}(\nabla^{\mathcal{W}}) \leq H := \mathrm{Stab}_G(s)$.

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- By definition, s corresponds to a P-equivariant map $\mathcal{G} \to \mathbb{W}$ or just as well a G-equivariant map $\underline{s} : \widehat{\mathcal{G}} \to \mathbb{W}$.

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- Since $\nabla^{\mathcal{W}}$ is a tractor connection, for any $x \in M$, the P-orbit $P \cdot \underline{s}(u) \subset \mathcal{O}$ is independent of the choice of $u \in \mathcal{G}_x$; this is the P-type.

C.o.d.: Main Theorem

- Unwinding definitions shows that the P-type of a point on the flat model is determined exactly by its H orbit, so we call the set of points with a given P-type a *curved orbit*. This gives the *curved orbit decomposition*

$$\boxed{M := \coprod_{a \in P \backslash \mathcal{O}} M_a}$$

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Theorem (Čap, Gover, Hammerl)

Given a Cartan geometry $(\mathcal{G}, \omega)$ of type (G, P) and a conjugacy class of groups $\mathrm{Hol}(\omega) \leq H \leq G$, we get the above decomposition, and each M_a is respectively equipped with a Cartan geometry of type (H, P_a), given by appropriate pullbacks via inclusions $\mathcal{G}_a \hookrightarrow \mathcal{G}|_{M_a}$.

Example: Orthogonal reduction redux

Suppose $h \in \Gamma(S^2\mathcal{V}^*)$ is nondegenerate of signature (p, q).

$$G := \mathrm{SL}(n+1, \mathbb{R}), \quad P := P_{\mathrm{SL}}, \quad H := \mathrm{Stab}_G(s) \cong \mathrm{SO}(p, q)$$

We know from understanding the reduction of the Klein geometry (G, P) to H that there are three curved orbits: $M_\pm$ inherit (pseudo-)Riemannian structures of signatures $(p-1, q)$, $(p, q-1)$, and M_0 inherits a conformal structure of signature $(p-1, q-1)$.

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- How to realize underlying structures explicitly?

Example (cont.): Constructing the geometric structure

Example ((Pseudo-)Riemannian structures)

An oriented projective structure comes equipped with a preferred section $X \in \Gamma(\mathcal{V} \otimes \mathcal{E}(1))$—identifiable with the canonical projection $J^1\mathcal{E}(1) \cong \mathcal{V}^* \to \mathcal{E}(1)$, and we can construct a canonical weighted object $\tau := h(X, X) \in \Gamma(\mathcal{E}(2))$. By construction, $M_+ = \{x : h(X, X)_x > 0\}$, and we can pick a connection $\nabla \in \mathbf{p}|_{M_+}$ s.t. $\nabla\tau = 0$. Then, the projective Schouten tensor P^∇ is nondegenerate because h is, and $\nabla^{\mathcal{V}} h = 0$ implies that $\nabla \mathsf{P}^\nabla = \frac{1}{n} \nabla \operatorname{Ric} = 0$, so ∇ is the Levi-Civita connection of the Einstein metric $g := \mathsf{P}^\nabla$.

Two comments

Remark (Compactification)

The fact that the metric g on M_+ and conformal structure $\mathbf{c}$ on M_0 both arise from h imposes strong compatibility conditions between the two structures. This leads to a notion of compactification of a (pseudo-)Riemannian metric with appropriate asymptotics by conformal geometry, all mediated via projective geometry.

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The map $\Pi^{S^2\mathcal{V}^*} : S^2\mathcal{V}^* \rightsquigarrow \mathcal{E}(2)$, $s \mapsto s(X, X)$, is the *BGG projection operator* associated to $\mathbb{W}$. For any $\mathbb{W}$ we get such a projection, and this map relates the holonomy reduction the geometry to an object that can usually be interpreted as data on the underlying manifold. If $\nabla^{\mathcal{W}} s = 0$, then $\Pi(s)$ is a solution to an invariant *BGG operator*.

Example: G_2 (case studied w Gover, Panai)

$$G := \mathrm{SL}(7,\mathbb{R}), \quad P := P_{\mathrm{SL}}, \quad H := G_2 \quad (\dim M = 6)$$

- $G_2 \hookrightarrow \mathrm{SO}(3,4)$ so this reduction refines the orthogonal one. Now, G_2 acts transitively on each of X_+, X_0, X_-, but we expect that the stronger reduction should determine more geometric structure on each orbit.
- G_2 reduction can be encoded by a parallel $\Phi \in \Gamma(\wedge^3 \mathcal{V}^*)$ of the appropriate G-type; determines $h_\Phi \in \Gamma(S^2\mathcal{V}^*)$.
- Object underlying Φ. is section of $\Lambda^2(T^*M(3))$. On M_+, trivializing with a power of $\tau > 0$ and raising an index gives a complex structure J. Then $\nabla^{\wedge^3\mathcal{V}^*}\Phi = 0$ implies that (g, J) defines a nearly Kähler structure on M_+.
- Similarly, on M_-, get a nearly para-Kähler structure.

Example: G_2 (cont.)

- On M_0, the conformal structure has holonomy contained in G_2; Nurowski showed these are $(2, 3, 5)$ conformal structures: These are induced canonically by distributions $\mathbf{D}$ satisfying $[\mathbf{D}, [\mathbf{D}, \mathbf{D}]] = TM_0$.

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- In this case, object underlying is a section $J_0 \in \Gamma(\mathrm{End}(TM_0) \otimes \mathcal{E}[1])$. Then, we recover $\mathbf{D} = \mathrm{im}(J_0) \otimes \mathcal{E}[-1]$, and $[\mathbf{D},\mathbf{D}] = \ker J_0$.

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- Two-out-of-three Rule: $\mathrm{U}(p', q')$ is the intersection of any two of (mutually compatible)

  $$\mathrm{SO}(2p', 2q'), \quad \mathrm{GL}(2m+2, \mathbb{C}), \quad \mathrm{Sp}(2m+2, \mathbb{R});$$

  can study these reductions separately and distill which features arise from which intermediate holonomy reductions; these three studied by Armstrong but only described briefly.

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  - $\mathrm{Sp}(2m+2, \mathbb{R})$: Underlying object: normal solution of Killing-type equation; induced geometry: (torsion-free) contact projective geometry

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  can study these reductions separately and distill which features arise from which intermediate holonomy reductions; these three studied by Armstrong but only described briefly.
  - $\mathrm{Sp}(2m+2, \mathbb{R})$: Underlying object: normal solution of Killing-type equation; induced geometry: (torsion-free) contact projective geometry
  - $\mathrm{GL}(2m+2, \mathbb{C})$: Underlying object: normal solution k of "adjoint BGG operator"; geometry (provided k is a projective symmetry, equiv., $W_{da}{}^{b}{}_{c}k^{d} = 0$): natural bundle with model fiber $U(1)$ over integrable c-projective structure.

Example: $\mathrm{U}(p', q')$ (cont.)

In fact, a reduction to $\mathrm{U}(p, q)$ (locally) implies a reduction to $\mathrm{SU}(p, q)$, which acts transitively on each of M_+, M_0, M_-; k is automatically a projective symmetry.

Example: $\mathrm{U}(p', q')$ (cont.)

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- On M_+, Sasaki-Einstein structure (dim $2m + 1$); locally fibers over Kähler-Einstein structure $(\bar{M}_+, \bar{g}, \bar{J})$ (dim $2m$) with model fibers $\mathrm{U}(1)$; c-projective structure is $(M_+, \bar{J}, [\nabla^{\bar{g}}])$

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- On M_0, Fefferman conformal structure of signature $(2p'-1, 2q'-1)$, i.e., induced on circle bundle over manifold $\bar{M}_0$ (dim $2m-1$) equipped with by hypersurface-type (in fact, integrable) CR structure with (nondegenerate) Levi-form $(p'-1, q'-1)$.

Example: $\mathrm{U}(p', q')$ (cont.)

In fact, a reduction to $\mathrm{U}(p, q)$ (locally) implies a reduction to $\mathrm{SU}(p, q)$, which acts transitively on each of M_+, M_0, M_-; k is automatically a projective symmetry.

- On M_+, Sasaki-Einstein structure (dim $2m + 1$); locally fibers over Kähler-Einstein structure $(\bar{M}_+, \bar{g}, \bar{J})$ (dim $2m$) with model fibers $\mathrm{U}(1)$; c-projective structure is $(M_+, \bar{J}, [\nabla^{\bar{g}}])$
- On M_0, Fefferman conformal structure of signature $(2p' - 1, 2q' - 1)$, i.e., induced on circle bundle over manifold $\bar{M}_0$ (dim $2m - 1$) equipped with by hypersurface-type (in fact, integrable) CR structure with (nondegenerate) Levi-form $(p' - 1, q' - 1)$.
- Downstairs (on barred objects) we get a compactification of Kähler-Einstein structures with appropriate asymptotics by CR structures—this is the classical one (can regard as mediated by *c*-projective geometry), which is hence compatible with the projective compactification.

Thank you.