Chains of path geometries on surfaces

Conformal Geometry, Analysis, and Physics:
Celebration of the work of Robin Graham on the occasion of his 65th birthday and academic retirement

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- G. Bor, W, Chains of path geometries on surfaces: theory and examples. arXiv:2201.09141

Definition (provisional)

A **path geometry** on a surface Σ is a family $\mathcal{P}$ of (unparametrized) curves on Σ such that for each point $p \in \Sigma$ and direction $\ell \in \mathbb{P}T_p\Sigma$ there is exactly 1 curve in $\mathcal{P}$ tangent to ℓ.

Example (A gallery of some (homogeneous) path geometries)

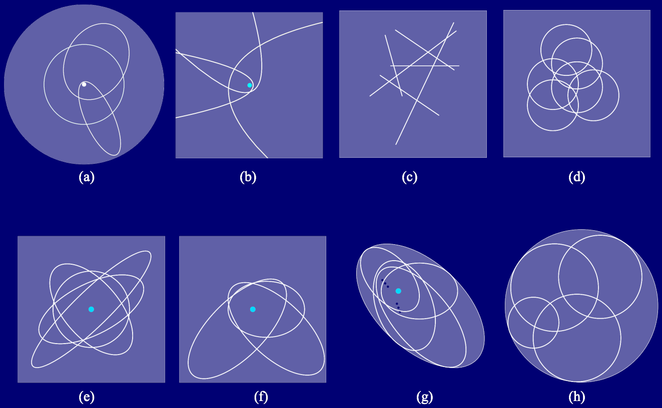

Figure: (a) Kepler ellipses of fixed major axis. (b) Kepler parabolas. (c) Straight lines. (d) Circles of fixed radius. (e) Hooke ellipses of fixed area. (f) Kepler ellipses of fixed minor axis. (g) Kepler ellipses tangent to a fixed Kepler ellipse. (h) Circles tangent to a fixed circle (horocycles).

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A **symmetry** of a path geometry $(\Sigma, \mathcal{P})$ is an equivalence of a path geometry with itself. A path geometry is **locally homogeneous** if its local symmetries act locally transitively.

Some ways to construct path geometries

- The geodesics of a linear connection ∇ (projective structure $[\nabla]$) constitute a path geometry. The path geometries that arise this way (locally) are called **projective**; most path geometries are not projective.

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- In particular, $\mathbb{P}_2$ equipped with the straight lines (i.e., images of punctured 2-dim. subspaces under the projection map $\mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}_2$) is the **flat model** of path geometry.

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  - A path geometry is **flat** if it is locally equivalent to $(\mathbb{P}_2, \{\text{straight lines}\})$. Example: (b), (c), (e) are flat.
- Given a path geometry $(\Sigma, \mathcal{P})$, we can define the **dual path geometry** $(\Sigma^*, \mathcal{P}^*)$, where Σ^* is the space of paths of and $\mathcal{P}^*$ is parametrized by Σ: The path in Σ^* corresponding to $p \in \Sigma$ is the space of paths in $\mathcal{P}$ passing through p. Example: (a) and (h) are dual; flat path geometries are self-dual (equivalent to their dual); (d) is self-dual and nonflat.

Structure on $\mathbb{P}T\Sigma$

- The 3-manifold $\mathbb{P}T\Sigma$ carries a canonical contact distribution H given by the *skating condition*: the point moves along the line, equivalently, the line rotates around the point. The fibers of $\mathbb{P}T\Sigma \to T\Sigma$ are integral curves of H, and we define the line field $\mathsf{E} \subset \mathsf{H}$. Any path $\gamma \in \mathcal{P}$ carries a tautological lift to $\mathbb{P}T\Sigma$ that lifts γ to its curve of tangents,

$$\widetilde{\gamma} := \{T_p\gamma : p \in \gamma\} \subset \mathbb{P}T\Sigma.$$

  The tangent lines to the lifted curves define a line field $\mathsf{F} \subset \mathsf{H}$ complementary to E, i.e.,

$$\boxed{\mathsf{H} = \mathsf{E} \oplus \mathsf{F}}.$$

Abstract definition of a path geometry

Definition

A **path geometry (on a surface)** is a triple $(M; \mathsf{E}, \mathsf{F})$ comprising a smooth 3-manifold M and line fields $\mathsf{E}, \mathsf{F} \subset TM$ together spanning a contact distribution. The **dual (path geometry)** of $(M; \mathsf{E}, \mathsf{F})$ is $(M; \mathsf{F}, \mathsf{E})$.

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By construction:

- Σ is the (local) space of integral curves of E, and
- Σ^* is the (local) space of integral curves of F.

$$\Sigma \longleftarrow M \longrightarrow \Sigma^*$$

The flat model reimagined

We can identify the space of lines in $\mathbb{P}_2$ with $\mathbb{P}_2^*$, so the underlying 3-manifold is $\mathbb{P}T\mathbb{P}_2 \cong \mathbb{P}T\mathbb{P}_2^* \cong V_{12}$, where

$$V_{12} = \{(p, \ell) \in \mathbb{P}_2 \times \mathbb{P}_2^* : p \in \ell\} \subset \mathbb{P}_2 \times \mathbb{P}_2^*.$$

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The complementary line fields E, E^* are the vertical subbundles of the projections $V_{12} \to \mathbb{P}_2$, $V_{12} \to \mathbb{P}_2^*$, respectively, that is, the flat model is $(V_{12}; \mathsf{E}, \mathsf{E}^*)$.

The flat model expressed as a correspondence space

The standard action of PSL_3 on $\mathbb{R}^3$ induces a transitive action on V_{12} that preserves the line fields, so $(V_{12}; \mathrm{E}, \mathrm{E}^*)$ is homogeneous, and in fact $\mathrm{Aut}(V_{12}; \mathrm{E}, \mathrm{E}^*) = \mathrm{PSL}_3$.

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The stabilizers of p, ℓ in PSL_3 are parabolic subgroups P_1, P_2, respectively, so we can regard the above as a correspondence space diagram:

$$\begin{array}{ccc} & \mathrm{PSL}_3 / (P_1 \cap P_2) & \\ \swarrow & & \searrow \\ \mathrm{PSL}_3 / P_1 & & \mathrm{PSL}_3 / P_2 \end{array} .$$

2nd-order o.d.e.s

- N.b. our newer definition is somewhat broader than the provisional one: It allows for paths through $p \in \Sigma$ to be defined only for an open subset of directions in $\mathbb{P}T_p\Sigma$.

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Example (2nd-order o.d.e.s)

Any 2nd-order o.d.e. $y'' = F(x, y, y')$ determines a path geometry

$$(\mathbb{R}^3_{xyp}; \mathrm{span}\{\partial_p\}, \mathrm{span}\{D_x\}),$$

where $\mathbb{R}^3_{xyp} = J^1(\mathbb{R}, \mathbb{R})$ and $D_x := \partial_x + p\partial_y + F\partial_p$. All $(M; \mathrm{E}, \mathrm{F})$ (locally) arise this way for some F. Here, $\Sigma = \mathbb{R}^2_{xy} = J^0(\mathbb{R}, \mathbb{R})$.

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$$\text{path geometries} \overset{\text{(locally)}}{\leftrightarrow} \begin{array}{l}\text{2nd-order o.d.e.s modulo} \\ \text{point transformations.}\end{array}$$

One o.d.e. (locally) realizing the flat model is $y'' = 0$ (i.e., $F(x, y, p) = 0$).

Chains on CR structures

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Chains are the exactly projections of the (nonvertical) null geodesics of (N, c) to M.

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For 3-dimensional path geometries the story is analogous: A Fefferman-type construction canonically assigns to a 3-dimensional path geometry $(M; \mathrm{E}, \mathrm{F})$

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Proposition (Nurowski–Sparling, '03)

For the path geometry determined by $y'' = F(x, y, y')$, the Fefferman conformal structure c on N is $[g]$, where

$$g = -dx(dp - F\,dx) + \frac{1}{6}(dy - p\,dx)[4F_p\,dx + F_{pp}(dy - p\,dx) - 4d\tau],$$

where τ is the standard coordinate on $\mathbb{R} \cong \mathrm{SO}(1,1)$.

Proposition (Bor–W)

The chains of the path geometry corresponding to a 2nd order o.d.e. $y'' = f(x, y, y')$ are the curves in $J^1(\mathbb{R}, \mathbb{R})$ which are the graphs of solutions $(y(x), p(x))$ of the system

$$\begin{aligned} y'' &= F + F_p\Delta + \frac{1}{2}F_{pp}\Delta^2 + \frac{1}{6}F_{ppp}\Delta^3 \\ p'' &= -\frac{2(p'-F)^2}{\Delta} + F_p(3p'-2F) + F_x + pF_y \\ &\quad + \left[F_{pp}(p'-F) + 2F_y\right]\Delta \\ &\quad + \frac{1}{6}\left[F_{ppp}(p'-2F) - F_{xpp} - pF_{ypp} + 4F_{yp})\right]\Delta^2 \end{aligned},$$

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where $\Delta := y' - p$.

Proof (sketch).

Compute null geodesics of g, use nullity to eliminate $\dot{\tau}$, then eliminate the parameter. □

A chain characterization of projective path geometries

Theorem (Bor–W)

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Proof (sketch).

The statement is local, so we may as well work with the path geometry defined by an o.d.e. $y'' = F(x, y, y')$, i.e., a function $F(x, y, p)$. It's well-known that such a path geometry is projective iff $\partial_p^4 F \equiv 0$. So, we must show that every solution $(y(x), p(x))$ of the geodesic system satisfies $F(x, y, y')$ iff F is a polynomial of degree ≤ 3 in p. But the right-hand side of the geodesic equation in y'' is just the cubic Taylor polynomial of $F(x, y, y')$ in p. □

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There is surely a less violent proof using the parabolic machinery concerning correspondence spaces and canonical curves (see Čap & Slovak, §§4.4, 5.3), and it may well generalize to other types of parabolic geometries.

Homogeneous path geometries

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Theorem (Tresse, 1896)

If the (local) symmetry group of a homogeneous path geometry is > 3, then it is locally equivalent to the flat model, and hence it is locally equivalent to a left-invariant path geometry on a Lie group.

In particular,

- we can locally specify a homogeneous path geometry up by the data $(\mathfrak{h}; \mathfrak{e}, \mathfrak{f})$, where $\mathfrak{h}$ is the Lie algebra of the (local) symmetry group and $\mathfrak{e}, \mathfrak{f} \subset \mathfrak{h}$ are 1-dimensional subspaces satisfying $[\mathfrak{e}, \mathfrak{f}] \not\subset \mathfrak{e} \oplus \mathfrak{f}$, and
- computing the chains amounts to computing null geodesics of a left-invariant metric in 4D.

Chains of the flat model

Goal: Identify the chains of the flat model.

We can realize the flat model as

$$\mathfrak{h}_3 := \left\{ \begin{pmatrix} \cdot & x^1 & x^3 \\ \cdot & \cdot & x^2 \\ \cdot & \cdot & \cdot \end{pmatrix} \right\}, \qquad \mathfrak{e} := \left\{ \begin{pmatrix} \cdot & * & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \right\}, \qquad \mathfrak{f} := \left\{ \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & * \\ \cdot & \cdot & \cdot \end{pmatrix} \right\}.$$

The affine plane $\left\{ \begin{pmatrix} * \\ * \\ 1 \end{pmatrix} \right\} \leftrightarrow \mathbb{R}^2$ is H_3-invariant, and H_3 acts freely and transitively on the incident pairs (q, ℓ) of points $q \in \mathbb{R}^2$ and nonvertical lines $\ell \subset \mathbb{R}^2$.

Conformal structure $\mathrm{c} = [g]$ on $G := H_3 \times \mathrm{SO}(1,1)$:

$$g = \theta^1\theta^2 + \frac{2}{3}\theta^3\theta^4, \qquad \theta^4\text{: a l.-i. form on } \mathrm{SO}(1,1)$$

We compute the null solutions of the Euler equations (geodesic flow on T^*G left-translated to $T_I^*G \cong \mathfrak{g}^*$).

The geodesic flow on T^*G projects via left translation to the Euler equations on $\mathfrak{g}^*$,

$$\boxed{\dot{P} = \mathrm{ad}^*_{A^{-1}P} P = \{H, P\}}, \qquad \text{where } h := \frac{1}{2}(P, A^{-1}P).$$

Chains of the flat model (cont.)

Inertia operator: $A : \mathfrak{g} \mapsto \mathfrak{g}^*$, $A = \begin{pmatrix} \cdot & 3 & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 \\ \cdot & \cdot & 2 & \cdot \end{pmatrix}$. Euler equation for $X = (x_1, \ldots, x_4) \in \mathfrak{g}$:

$$\dot{x}^1 = \frac{2}{3} x^1 x^4, \qquad \dot{x}^2 = -\frac{2}{3} x^2 x^4, \qquad \dot{x}^3 = \dot{x}^4 = 0.$$

The general null $(H = 0)$ solution is

$$x^1 = ae^{ct}, \quad x^2 = be^{-ct}, \quad x^3 = -\frac{ab}{c}, \quad x^4 = \frac{3c}{2}.$$

So, for a null geodesic $g = \begin{pmatrix} 1 & z & y \\ \cdot & 1 & x \\ \cdot & \cdot & 1 \end{pmatrix}_s : \mathbb{R} \to G$, $X := g^{-1}\dot{g} : \mathbb{R} \to \mathfrak{g}$ satisfies

$$\dot{x} = x^2 = be^{-ct}, \quad \dot{y} - \dot{x}z = x^3 = -\frac{ab}{c}, \quad \dot{z} = x^1 = ae^{ct}.$$

By l.-i. we may as well take $g(0) = I \in H_3$, and solving explicitly gives that (x, y) traces a straight line in $\mathbb{R}^2$.

Interpreting the previous characterization gives:

Proposition (Bor–W)

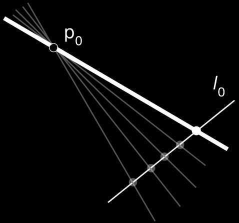
1. *For any non-incident* $(p_0, \ell_0) \in \mathbb{P}_2 \times \mathbb{P}_2^*$ *(i.e.,* $p_0 \notin \ell_0$*, equivalently,* $(p_0, \ell_0) \notin V_{12}$*), the locus*

$$\{(p, \ell) \in V_{12} : p \in \ell_0, p_0 \in \ell\}$$

   is a chain of the flat model $(V_{12}, \mathrm{E}, \mathrm{E}^*)$*, and*

2. *all chains arise as such loci; in particular there is a bijection*

$$\{chains\ on\ V_{12}\} \leftrightarrow (\mathbb{P}_2 \times \mathbb{P}_2^*) \setminus V_{12}.$$



Hooke ellipses

Recall that the **Hooke ellipses** are the ellipses in $\mathbb{R}^2$ of area (say) π centered at the origin
Hooke ellipses:

$$H := \mathsf{SL}_2, \quad \mathfrak{e} := \left\{\begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix}\right\}, \quad \mathfrak{f} := \left\{\begin{pmatrix} \cdot & -1 \\ 1 & \cdot \end{pmatrix}\right\}.$$

Realizable by $y'' = (xy' - y)^3$.

We can proceed as before to find the chains on H. Since $(H; \mathsf{E}, \mathsf{F})$ is projective, the projections of the chains to $\Sigma = \mathbb{R}^2 \setminus \{(0,0)\}$ are just the Hooke ellipses.

Horocycles vis-à-vis Hooke ellipses

Recall that **horocycles** are circles (interior-)tangent to a fixed circle, say, the boundary $\partial\mathbb{D}$ of $\mathbb{D}$.

$$H := \mathrm{SL}_2, \quad \mathfrak{e} := \left\{\left(\begin{smallmatrix} \cdot & -1 \\ 1 & \cdot \end{smallmatrix}\right)\right\}, \quad \mathfrak{f} := \left\{\left(\begin{smallmatrix} \cdot & 1 \\ \cdot & \cdot \end{smallmatrix}\right)\right\}.$$

Realizable by $y'' = \frac{y' + [1+(y')^2]^{3/2}}{x}$.

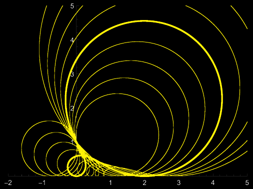
This path geometry is dual to that of the Hooke ellipses, so:

- The set of Hooke ellipses passing through a fixed point $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ can be canonically identified with a horocycle.
- The set of horocycles passing through a fixed point $z \in \mathbb{D}$ can be canonically identified with a Hooke ellipse.
- The projections of the chains of the horocycles can be obtained by projecting the chains of the Hooke ellipses in the other direction.

Projections of chains of horocycles

$$\mathrm{SL}_2 \to \mathbb{R}^2 \setminus \{(0,0)\}, \qquad \mathrm{SL}_2 \to \mathbb{D}$$

The horocycle path geometry is not projective, so its chains are not horocycles. Instead, in the upper half-plane model:



$$(x^2+y^2)^2-[4cx+(c^2+4)y](x^2+y^2)+(6c^2-2)x^2+2c^3xy+6y^2 \\ -4c(c^2-1)x-(c^4-3c^2+4)y+(c^2-1)^2=0$$

These curves are examples of rational *bicircular quartics* (inversions of conics w.r.t. circles), studied by Casey in the 1870s.

Some further questions:

- Is there a purely geometric characterization of the projections of the horocycle chains to $\mathbb{D}$?
- Is there a geometric characterization of the projections of chains of circles of fixed radius?
- Do the chains of any other homogeneous path geometries have interesting projections?
- For higher dimensions, the appropriate generalization of path geometries on surfaces is to Lagrangean contact structures; what can we say in higher dimensions?
- What are the analogues of the chain characterization of projective path geometries to other correspondence constructions, esp. in the setting of canonical curves in parabolic geometry?

Thank you.

Congratulations and happy birthday to Robin Graham!